International Journal of Theoretical Physics, Vol. 45, No. 7, July 2006 (© 2006)
DOI: 10.1007/s10773-006-9059-6

Equations Holding in Hilbert Lattices

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Received July 18, 2005; accepted January 26, 2006
Published Online: June 6, 2006

We produce and study several sequences of equations, in the language of orthomodular lattices, which hold in the ortholattice of closed subspaces of any classical Hilbert space, but not in all orthomodular lattices. Most of these equations hold in any orthomodular lattice admitting a strong set of states whose values are in a real Hilbert space. For some of these equations, we give conditions under which they hold in the ortholattice of closed subspaces of a generalised Hilbert space. These conditions are relative to the dimension of the Hilbert space and to the characteristic of its division ring of scalars. In some cases, we show that these equations cannot be deduced from the already known equations, and we study their mutual independence. To conclude, we suggest a new method for obtaining such equations, using the tensorial product.

KEY WORDS: Hilbert lattices; orthomodular lattices; equations; quantum logic; quantum computation.

PACS numbers: 02.10, 03.65, 03.67

1. INTRODUCTION

If $\mathcal{H}$ is a classical Hilbert space, a subspace M of $\mathcal{H}$ is topologically closed iff it is orthogonally closed, i.e., if it coincides with its biorthogonal. Let us denote by $\mathcal{C}(\mathcal{H})$ the ortholattice of the closed subspaces of a Hilbert space $\mathcal{H}$.

The variety of orthomodular lattices (OMLs) is an algebraic generalization of the class of ortholattices of the form $\mathcal{C}(\mathcal{H})$. This variety is obtained by adding only one new equation to those of ortholattices, the orthomodularity. This equation is very powerful and allows to extend to general OMLs many properties relative to ortholattices of the form $\mathcal{C}(\mathcal{H})$, and also to extend the definitions of mathematical entities, such as states and observables, useful in the Hilbert space approach to quantum mechanics.

But it was known for a long time that there are equations holding in otholattices of the form $\mathcal{C}(\mathcal{H})$ but not in general OMLs (see Godowski, 1981; Godowski and Greechie, 1984; Mayet, 1985, 1986). More recently (Megill and Pavičić,

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2000), some more results have been published, motivated by the possible repercussions to quantum computing problems.

In the following, the set of all equations holding in all OMLs of the form $\mathcal{C}(\mathcal{H})$, but not in every OML, will be denoted by $\mathcal{E}$.

In Section 3, we recall the different known results about $\mathcal{E}$, with some comments. The rest of the paper is devoted to several sequences of new equations in $\mathcal{E}$. In fact, some of them, in particular those of Theorem 4.1 were already presented and studied in Mayet (1987), but have never been published elsewhere. As concern these equations, we study problems of independence and we show that some of them, when interpreted in generalized Hilbert lattices, may be regarded as conditions on the dimension together with the characteristic of the underlying division ring.

2. ORTHOMODULAR LATTICES AND GENERALIZED HILBERT LATTICES

Let us recall some basic facts of the theory of orthomodular lattices. For more details the reader may consult Kalmbach (1983) or Pták, and Pulmannová (1990).

An OML is an algebra $(L, 0, 1, \vee, \wedge, ^{\perp})$, where $(L, 0, 1, \vee, \wedge)$ is a bounded lattice, and $^{\perp}$ is an antitone (i.e., such that $a \leq b$ implies $b^{\perp} \leq a^{\perp}$) and involutive unary operation so that $a \vee a^{\perp} = 1$ (which implies $a \wedge a^{\perp} = 0$), and the orthomodular law $a \vee b = a \vee (a^{\perp} \wedge (a \vee b))$ holds true. The class of OMLs is a variety which contains the variety of Boolean algebras. Two elements a, b of an OML are said to be orthogonal ($a \perp b$) if $a \leq b^{\perp}$ (or, equivalently, if $b \leq a^{\perp}$). For any a, b in an OML, we will use the notation $a \to b$ for $a^{\perp} \vee (a \wedge b)$.

If a, b are two elements of an OML, one says that a and b commute if $a = (a \wedge b) \vee (a \wedge b^{\perp})$, or equivalently if $b = (b \wedge a) \vee (b \wedge a^{\perp})$. A triple (a, b, c) of elements of an OML such that one of these elements commutes with both two others is distributive, which means that, for any permutation (u, v, w) of (a, b, c), $u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$ and $u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$.

If D is a subset of an OML whose any two elements commute, the sub-OML generated by D is a Boolean algebra. A block of an OML is a maximal Boolean subalgebra. Any finite OML L may be represented by its Greechie diagram (Kalmbach, 1983), a hypergraph, whose vertices correspond to the atoms and whose edges represent the blocks of L. Here we will use such diagrams only in the simplest case described in Greechie (1971).

The following results about orthomodular spaces and generalized Hilbert lattices can be found in Piron (1963); Varadowajan (1984); Keller (1985); Grass and Künzi) (1985); Soler (1995); Holland (1995).

Let K be a division ring equipped with an involutive anti-automorphism denoted by *, and let $\mathcal{H}$ be a left vector space over K. A Hermitian form on $\mathcal{H}$ is a mapping $\langle ., . \rangle$ from $\mathcal{H} \times \mathcal{H}$ to K such that:

(a) for any y in $\mathcal{H}$, the mapping $x \mapsto \langle x, y \rangle$, from $\mathcal{H}$ to K is linear;
(b) for all x, y in $\mathcal{H}$, $\langle y, x \rangle = \langle x, y \rangle^*$;
(c) if $x \in \mathcal{H}$ is such that $\langle x, y \rangle = 0$ for any $y \in \mathcal{H}$, then $x = 0$.

The vector space $\mathcal{H}$, when equipped with a Hermitian form, is called a Hermitian space. Two vectors x, y of the Hermitian space $(\mathcal{H}, \langle ., . \rangle)$ are said to be orthogonal, which is denoted $x \perp y$, if $\langle x, y \rangle = 0$. For any subset S of $\mathcal{H}$, the set $\{x \in \mathcal{H} : \forall y \in S, x \perp y\}$, is always a subspace of $\mathcal{H}$, which is denoted by $S^\perp$. A subspace M of $\mathcal{H}$ is called closed if it is of the form $S^\perp$ or equivalently if $M = M^{\perp\perp}$. Every finite dimensional subspace is closed. The Hermitian space $(\mathcal{H}, \langle ., . \rangle)$ is called orthomodular if, for any closed subspace M, $\mathcal{H}$ is the direct sum of M and $M^\perp$: $\mathcal{H} = M \oplus M^\perp$, and this implies that the Hermitian form is anisotropic: $\langle x, x \rangle = 0$ implies $x = 0$. In this case, $\langle \mathrm{x, y} \rangle$ is called the scalar product of x and y.

Let $(\mathcal{H}, \langle ., . \rangle)$ be an orthomodular space (also called generalized Hilbert space) over K. Then the set $\mathcal{C}(\mathcal{H})$ of all closed subspaces of $\mathcal{H}$, when ordered by inclusion and equipped with the involution $M \mapsto M^\perp$, is a complete, atomic, irreducible orthomodular lattice satisfying the covering law (Piron 1963), in which the meet and join of two elements are defined by: $M \wedge N = M \cap N$ and $M \vee N = (M + N)^{\perp\perp}$. An ortholattice $\mathcal{L}$ isomorphic to such an OML $\mathcal{C}(\mathcal{H})$ is called a *generalized Hilbert lattice* (GHL). In the particular case where $\mathcal{H}$ is a classical Hilbert space over **R**, **C**, or **H** respectively the field of real numbers, the field of complex numbers and the skew field of quaternions, endowed with their natural conjugations), $\mathcal{H}$ is always orthomodular, and then any OML isomorphic to $\mathcal{C}(\mathcal{H})$ is called a *classical Hilbert lattice* (HL). Notice that in Section 1 above, we deal only with classical Hilbert lattices.

Any orthomodular lattice $\mathcal{L}$, of height at least 4, satisfying the above four properties (complete, atomic, irreducible, satisfying the covering law), is a GHL.

In the finite-dimensional case, and for any division ring K, $(\mathcal{H}, \langle ., . \rangle)$ is orthomodular iff the Hermitian form $\langle ., . \rangle$ is anisotropic and in this case it is quite easy to construct nonclassical orthomodular spaces. But this condition of anisotropy is not sufficient in the infinite dimensional case. However, it has been shown (Keller, 1980), (Grass, *et al.*, 1985) that there exist many infinite-dimensional nonclassical orthomodular spaces. In particular, for any characteristic of the underlying division ring K, there are examples of nonclassical orthomodular spaces of any finite-dimension, and also of infinite dimension (Grass, *et al.*, 1985).

If $\mathcal{H}$ is any orthomodular space and if $M \in \mathcal{C}(\mathcal{H})$, since $H = M \oplus M^\perp$, every $x \in H$ has a unique representation of the form $x = x_1 + x_2$, with $x_1 \in M$ and $x_2 \in M^\perp$, and this allows to define the (orthogonal) projection mapping $pr_M : \mathcal{H} \mapsto M$ by $pr_M(x) = x_1$, which is obviously linear. It is easily seen that if $M_1, \ldots, M_k \in \mathcal{C}(\mathcal{H})$ are mutually orthogonal, then for any $x \in M_1 \vee \cdots \vee M_n$, $x = pr_{M_1}(x) + \cdots pr_{M_n}(x)$.

Solèr proved in (1995) the following outstanding result:
An infinite-dimensional orthomodular space over K is a classical Hilbert space if and only if it contains a γ-orthogonal system, where γ is a nonzero element of K, that is a sequence $(e_n)_{n\in N}$ of pairwise orthogonal vectors such that, for any $n \in \mathbf{N}$, $\langle e_n, e_n\rangle = \gamma$.

This shows in particular that in a nonclassical orthomodular space H, if x is any nonzero vector, there is generally no vector $u \in Kx$ such that $\langle u, u\rangle = 1_K$, and that if y is a nonzero vector orthogonal to x there is generally no $v \in Ky$ such that $\langle v, v\rangle = \langle x, x\rangle$.

3. ORTHOARGUESIAN EQUATIONS AND EQUATIONS RELATED TO STATES

All the equations we are dealing with here are equations in the theory of OMLs.

In the theory of OMLs, any inequality $a \leq b$ is obviously equivalent to the identity $a = a \wedge b$. Moreover, it is sometime useful to write an equation under the form of an implication, as follows: if $x_1, \ldots, x_n$ are n variables, $E(x_1, \ldots, x_n)$ an equation and $I \subseteq \{1, \cdots n\} \times \{1, \ldots, n\}$, then the formule: $(\forall (i, j) \in I, x_i \perp x_j) \Rightarrow E(x_1, \ldots, x_n)$ is equivalent to an equation (cf. Mayet, 1986, Lemma 1).

Let E and E' be two equations, and let $\mathcal{F}$ be a set of equations. The equation E is called a consequence of $\mathcal{F}$ if E holds in any OML in which each equation in $\mathcal{F}$ holds, otherwise E is said to be independent of $\mathcal{F}$. We say that E is stronger than E' (and that E' is weaker than E) if E' is a consequence of $\{E\}$. If E is stronger than E', and if E' is not stronger than E, we say that E is strictly stronger than E', and that E' is stricly weaker than E. If each of the two equations E, E' is stronger than the other one, then these two equations are said to be equivalent.

We recall that $\mathcal{E}$ denotes the set of equations which hold in any classical Hilbert lattice, but not in all OMLs.

The first equation in $\mathcal{E}$ was found in 1975 by A. Day (unpublished). It is the orthoarguesian equation denoted by OA:

$$(a_i \perp b_i,\ i = 1, 2, 3) \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_3 \vee b_3) \leq b_1 \vee (a_1 \wedge (a_2 \vee [t_{1,2} \wedge (t_{1,3} \vee t_{2,3})])) \qquad \text{(OA)}$$

where $t_{i,j} = (a_i \vee a_j) \wedge (b_i \vee b_j)$.
Since that time, several related equations in $\mathcal{E}$ have been obtained in a similar way. The first one, denoted here by OA' (Godowski, 1984) is strictly weaker than OA:

$$(a_i \perp b_i,\ i = 1, 2) \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq b_1 \vee (a_1 \wedge (a_2 \vee t_{1,2})) \qquad \text{(OA}')$$

More recently, Megill and Pavičić (2000), have obtained more equations of the same family, and in particular an infinite sequence $(n\text{OA})_{n\geq 3}$ of generalized orthoarguesian equations such that 3OA and 4OA are, respectively, the above equations OA' and OA.
The proof of the fact that an equation of this family holds in all HLs uses essentially the decomposition of a vector on orthogonal subspaces together with basic applications of the associativity and commutativity of the addition of vectors, and the inclusion $M + N \subseteq M \vee N = (M + N)^{\perp\perp}$ for any M, N in a HL.

Let us recall, for instance, how it can be proved that OA' holds in any HL. Let us assume that a_i and b_i, $i = 1, 2$ are elements of a HL $\mathcal{L}$ such that $a_i \perp b_i$. Let $x \in (a_1 \vee b_1) \wedge (a_2 \vee b_2)$. If for $i = 1, 2$, we define $x_i = pr_{a_i}(x)$ and $y_i = pr_{b_i}(x)$, then $x = x_1 + y_1 = x_2 + y_2$. Observing that $x_1 = x_2 + (y_2 - y_1)$, and that $y_2 - y_1 = x_1 - x_2 \in (a_1 \vee a_2) \wedge (b_1 \vee b_2) = t_{1,2}$, we obtain that $x = x_1 + y_1 \in b_1 \vee (a_1 \wedge (a_2 \vee t_{1,2}))$, and it follows that OA' holds true.

The key of this proof is that the relations $x_1 = x_2 + z$ with $z = y_2 - y_1 = x_1 - x_2$ has allowed us to transform the obvious initial equation:

$$(a_i \perp b_i,\ i = 1, 2) \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq b_1 \vee a_1$$

into OA' by substituting to the unique instance of the variable a_1 in the right hand side of the inequality, the term $a_1 \wedge (a_2 \vee t_{1,2})$. We observe that, using the relation $x_1 = y_2 + z'$, where $z' = x_2 - y_1 = x_1 - y_2$ (or $x_1 = -y_1 + x$) we can also replace an instance of a_1 on the right hand side of the inequality by $a_1 \wedge (b_2 \vee ((a_1 \vee b_2) \wedge (a_2 \vee b_1)))$ (or by $a_1 \wedge (b_1 \vee ((a_1 \vee b_1) \wedge (a_2 \vee b_2)))$, respectively).

Now, let us suppose the supplementary hypothesis $a_3, b_3 \in \mathcal{L}$, with $a_3 \perp b_3$, and assume that $x \in (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_3 \vee b_3)$. Then $x = x_3 + y_3$, where $x_3 = pr_{a_3}(x)$ and $y_3 = pr_{b_3}(x)$. We observe that the vector $z = y_2 - y_1 = x_1 - x_2$ can be written $z = (y_2 - y_3) + (y_3 - y_1)$ where $y_2 - y_3 = x_3 - x_2 \in t_{2,3}$ and $y_3 - y_1 = x_1 - x_3 \in t_{1,3}$. This show that, starting from the equation:

$$(a_i \perp b_i, i = 1, 2, 3) \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_3 \vee b_3) \leq b_1 \vee (a_1 \wedge (a_2 \vee t_{1,2}))$$

which is an obvious consequence of OA', and replacing the term $t_{1,2}$ on the right-hand side of the inequality by $t_{1,2} \wedge (t_{1,3} \vee t_{2,3})$, we obtain that equation OA holds true. For each $n \geq 1$, equation OA_n can be deduced from OA_{n-1} in a similar way (cf. Megill and Pavičić, 2000).

In the above proofs, starting from an obvious inequality, we have carried out some substitutions on the right-hand side of this inequality. Each of these substitutions is justified by some basic calculations using only the associativity and the commutativity of the addition of vectors. In fact, we may carry out any finite number of such substitutions: in each case, we obtain an equation holding not only in all HL, but more generally in any GHL, since in a GHL the proof is exactly the same. For instance, if we replace, in the term of the left-hand side of OA, some occurrences of a_1 by $a_1 \wedge (a_3 \vee ((a_1 \vee a_3) \wedge (b_1 \vee b_3)))$, the new equation obtained holds in every GHL.

Moreover, this may be generalized by using decompositions of an orthomodular space into direct sums of n pairwise orthogonal closed subspaces, with $n \geq 2$. Let us illustrate with a simple example this general method.

Let us start from the obvious relation:

$$(a_1 \perp b_1,\ a_2 \perp b_2,\ b_2 \perp c_2,\ a_2 \perp c_2) \ \Rightarrow\ (a_1 \vee b_1) \wedge (a_2 \vee b_2 \vee c_2) \ \leq\ a_1 \vee b_1$$

If we imagine that a_1, b_1, a_2, b_2, c_2 are elements of a HL (or of a GHL) satisfying the above relations of orthogonality, and that $x \in (a_1 \vee b_1) \wedge (a_2 \vee b_2 \vee c_2)$, then $x = x_1 + y_1 = x_2 + y_2 + z_2$, where $x_i = p_{a_i}(x)$, $y_j = p_{b_j}(x)$ and $z_2 = p_{c_2}(x)$. The relations $y_1 = x_2 + (y_2 + z_2 - x_1) = y_2 + (x_2 + z_2 - x_1) = z_2 + (x_2 + y_2 - x_1)$ allow us (for instance) to replace the first instance of b_1 on the right-hand side of the inequality, successively by $b_1 \wedge t_1, b_1 \wedge t_2$, and $b_1 \wedge t_3$, where $t_1 = a_2 \vee ((a_1 \vee b_2 \vee c_2) \wedge (a_2 \vee b_1))$, $t_2 = b_2 \vee ((a_1 \vee a_2 \vee c_2) \wedge (b_1 \vee b_2))$, and $t_3 = c_2 \vee ((a_1 \vee a_2 \vee b_2) \wedge (b_1 \vee c_2))$. In this way, we obtain the following equation:

$$(a_1 \perp b_1,\ a_2 \perp b_2,\ b_2 \perp c_2,\ a_2 \perp c_2) \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2 \vee c_2) \leq\ a_1 \vee (b_1 \wedge t_3 \wedge t_2 \wedge t_1)$$

which holds in any GHL. By setting $c_2 = 0$ in this equation, we obtain an equation obviously stronger than OA', which proves that this equation belongs to $\mathcal{E}$. Then the problem is to compare this equation with other equations obtained by this method.

In short, this general method allows to obtain very simply a lot of new equations in $\mathcal{E}$, since for each new substitution we obtain a stronger equation belonging to $\mathcal{E}$. We will denote by $\mathcal{E}_0$ the set of all equations in $\mathcal{E}$ obtained by applying this method. Each equation in $\mathcal{E}_0$ holds in any GHL. Unfortunately, the problem of the hierarchy between these equations seems to be very difficult. In Megill and Pavičić, 2000, the authors have shown that OA_2 is strictly stronger than OA by using massive calculations by computer.

We will see in Section 4 that a slight generalisation of this method allows to obtain significant equations in $\mathcal{E}$, which are easier to study.

A real-valued state on an OML $\mathcal{L}$ is a mapping s from $\mathcal{L}$ to the real closed interval [0, 1] such that $s(1_{\mathcal{L}}) = 1$, and for any $a, b \in \mathcal{L}$ such that $a \perp b$, $s(a \vee b) = s(a) + s(b)$. The OML $\mathcal{L}$ admits a strong (or rich) set of real-valued states if, for any elements a, b of $\mathcal{L}$ such that $a \not\leq b$, there exists a real-valued state s on $\mathcal{L}$ such that $s(a) = 1$ and $s(b)\langle 1$. Any HL admits a strong set of real-valued states: in a HL, if $a \not\leq b$, and if u is a unit vector in $a \setminus b$, then the mapping s_u defined, for $c \in \mathcal{L}$ by $s_u(c) = \langle u, pr_c(u)\rangle = \langle pr_c(u), pr_c(u)\rangle$ is a real-valued state (called a "pure state") such that $s_u(a) = 1$ and $s_u(b)\langle 1$.

Godowski (1981), starting from a sequence of finite OMLs without a strong set of real-valued states, discovered a sequence $(G_n)_{n \geq 3}$ of equations in $\mathcal{E}$ such that, for each n, G_{n+1} is strictly stronger than G_n. For each $n \geq 3$, the equation G_n may be written as follows:

$$a_1 \perp a_2 \perp a_3 \perp \cdots \perp a_{2n} \perp a_1 \quad \Rightarrow$$
$$(a_1 \vee a_2) \wedge (a_3 \vee a_4) \wedge \cdots \wedge (a_{2n-1} \vee a_{2n}) \ \leq a_{2n} \vee a_1 \qquad (G_n)$$

In Mayet (1986), the result of Godowski was generalized into a general method, allowing to obtain by an effective procedure, for each OML $\mathcal{L}$ whithout a strong set of real-valued states, an equation holding in any OML with a strong set of real-valued states, and failing in $\mathcal{L}$. As any HL admits a strong set of real-valued states, the equations obtained in this way all belong to $\mathcal{E}$. In Mayet (1986) were given some examples for illustrating the method, but the corresponding equations where shown by Megill and Pavičić (2000) to be consequences of Godowski's equations. These authors have even expressed some doubts about the existence of equations obtained by this method that are not consequences of those of Godowski, but they report that, since then, they have obtained such equations (unpublished).

4. A SEQUENCE OF NEW EQUATIONS

Let $\mathcal{L} = \mathcal{C}(\mathcal{H})$ be any GHL, let $n \geq 3$ be an integer, and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of $\mathcal{L}$ satisfying the following set of conditions, denoted by (Ω):

$$\forall i, j \in \{1, \ldots, n\},\ i \neq j,\ a_i \perp aj, \text{ and } \forall i \in \{1 \cdots n\}, a_i \perp bi \qquad (\Omega)$$

To be short, let us define:

$$a = a_1 \vee \cdots \vee a_n \qquad b = b_1 \vee \cdots \vee b_n \qquad q = (a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n).$$

Let $x \in a \wedge q$. Let us define, for $i = 1 \cdots n$, $x_i = pr_{a_i}(x)$ and $y_i = pr_{b_i}(x)$. Then

$$x = x_1 + y_1 = \cdots = x_n + y_n = x_1 + \cdots + x_n.$$

and it follows that $(n-1)x = y_1 + \cdots + y_n$. Hence, if $(n-1)1_K \neq 0_K$, in other words if the characteristic of the underlying division ring K is not a divisor of $n-1$, we conclude that $x \in b = b_1 \vee \cdots \vee b_n$, which proves that $a \wedge q \leq b$.

Now, let us assume that $(n-1)1_K = 0_K$, and let us separate two cases.

a) Let us suppose that the orthomodular space $\mathcal{H}$ is of dimension $\leq n-1$. Then, since the subspaces $a_1, \cdots a_n$ are pairwise orthogonal, there exists $i \in \{1, \ldots, n\}$ such that $a_i = \{0\}$. Therefore, $q \leq b_i \leq b$, and it follows that the relation $a \wedge q \leq b$ holds true.
b) On the other hand, let us suppose that the dimension of $\mathcal{H}$ is at least n. Let $u_1, \ldots, u_n$ be n pairwise orthogonal vectors in $\mathcal{H}$, and, for $i = 1 \cdots n$, let a_i and b_i be the 1D subspaces of $\mathcal{H}$ generated by u_i and $v_i = \sum_{j \neq i} u_j$, respectively.

Then $u = u_1 + \cdots + u_n$ is a nonzero vector in $a \wedge q$. Let us suppose that $u \in b$. As $b_1, \ldots, b_n$ are 1D, $b = b_1 + \cdots + b_n$, hence there exists $\lambda_1, \ldots, \lambda_n \in K$ such that, if we define $\lambda = \lambda_1 + \cdots + \lambda_n$,

$$u = \lambda_1 v_1 + \cdots + \lambda_n, v_n = (\lambda - \lambda_1)u_1 + \cdots + (\lambda - \lambda_n)u_n$$

hence, by the unicity of the representation of u as a linear combination of the vectors $u_1, \ldots, u_n$, which are linearly independent,

$$\lambda_1 = \cdots = \lambda_n = \lambda - 1_K.$$

It follows that $\lambda = \lambda_1 + \cdots \lambda_n = n(\lambda - 1_K) = (n-1)(\lambda - 1_K) + \lambda - 1_K = \lambda - 1_K$, and we obtain $1_K = 0_K$, a contradiction.
This proves that the relation $a \wedge q \leq b$ fails in $\mathcal{L}$.
Let us summarize these results in the following Theorem.

Theorem 4.1. *Let $n \geq 3$ be an integer, let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be $2n$ variables and let (Ω) be the set of conditions of orthogonality: for $i, j \in \{1, \ldots, n\}$, $i \neq j$, $a_i \perp a_j$, and for $i = 1, \ldots, n$, $a_i \perp b_i$.*
Let us define the terms $a = a_1 \vee \cdots \vee a_n$, $q = (a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n)$, and $b = b_1 \vee \cdots \vee b_n$.
Let us denote by E_n the equation:

$$(\Omega) \Rightarrow a \wedge q \leq b \qquad (E_n)$$

Let $\mathcal{H}$ be any orthomodular space, and K be its scalar division ring.
Then the following two statements are equivalent:

(i) the equation E_n holds in $\mathcal{C}(\mathcal{H})$;
(ii) the dimension of H is at most $n-1$, or $(n-1)1_K \neq 0_K$.

In particular, E_n holds in any classical Hilbert lattice.

The fact that these equations E_n does not hold in every GHL shows that they are not of the same kind as those studied in Section 2. More precisely we have the following obvious result:

Theorem 4.2. *For any integer $n \geq 3$ the equation E_n is not a consequence of the set $\mathcal{E}_0$ of all equations obtained by the general method described in Section 3, even if we add the modularity. In particular, E_n cannot be deduced from the orthoarguesian law and all its generalizations.*

Proof. Let $n \geq 3$, let k be a prime divisor of $n-1$, and let H be an orthomodular space of dimension n over a field K of characteristic k. Then $(n-1)1_K = 0_K$, hence the equation E_n fails in $\mathcal{C}(\mathcal{H})$ while any equation in $\mathcal{E}_0$ holds in this OML, and the modular law too. □

Equation E_n may be written when $n = 2$, but then it is without any interest since it holds in any OML. Indeed, if a_1, a_2, b_1, b_2 are any elements of an OML such that $a_1 \perp a_2, a_1 \perp b_1$ and $a_2 \perp b_2$, then, for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$, since a_j and b_i both commute with a_i, we have, by a special case of distributivity in OMLs, $(a_i \vee a_j) \wedge (a_i \vee b_i) = a_i \vee c_i$, where $c_i = a_j \wedge b_i$. It follows that $(a_1 \vee a_2) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) = (a_1 \vee c_1) \wedge (a_2 \vee c_2)$. Then, since any two elements of $\{a_1, a_2, c_1, c_2\}$ commute, there exists a block containing $\{a_1, a_2, c_1, c_2\}$, and by easy Boolean calculations, we obtain $(a_1 \vee c_1) \wedge (a_2 \vee c_2) = c_1 \vee c_2 \leq b_1 \vee b_2$.

The method used to prove that equation E_n holds in any GHL is very close to the method described in Section 3, and used for orthoarguesian equations: we have decomposed the vector $x \in a \wedge q$ into the sum of its projections onto mutually orthogonal subspaces, in different ways; after some calculations (a little less basic than in the first case) we have obtained that $x \in b_1 + \cdots + b_n \subseteq b_1 \vee \cdots \vee b_n = b$.

Now, we will see that these equations can also be obtained by a method using Hilbert-space-valued states.

5. HILBERT-SPACE-VALUED STATES

Let $\mathcal{L}$ be any OML. By a Hilbert-space-valued state (H-state) on $\mathcal{L}$, we mean a mapping s from $\mathcal{L}$ to a classical Hilbert-space $\mathcal{H}$, such that $||s(1_{\mathcal{L}})|| = 1$ (where $||.||$ is the norm defined as usual on $\mathcal{H}$ by $||x|| = \sqrt{\langle x, x\rangle}$) and, for any $a, b \in \mathcal{L}$,

$$a \perp b \Rightarrow s(a) \perp s(b) \quad \text{and} \quad s(a \vee b) = s(a) + s(b).$$

According to whether the underlying field of $\mathcal{H}$ is **R**, **C** or **H**, the H-state s is called a RH-state, a CH-state, or a QH-state, respectively. We say that an OML $\mathcal{L}$ admits a strong set of RH-states if there exists a real Hilbert-space $\mathcal{H}$ such that, for any two elements a, b in $\mathcal{L}$ satisfying the condition $a \not\leq b$, there exists a $\mathcal{H}$-valued state s on $\mathcal{L}$ such that $||s(a)|| = 1$ and $||s(b)||\langle 1$. We have similar definitions by replacing the field **R** by **C** or **H**.

Let $(\mathcal{H}, \langle ., .\rangle)$ be a complex Hilbert-space. Then $\mathcal{H}$ can be considered as a vector space $\mathcal{H}'$ over **R**, and then, when equipped with the scalar product $\langle ., .\rangle'$ defined by $\langle x, y\rangle' = R(\langle x, y\rangle)$ (where $R(\lambda)$ denotes the real part of the complex number λ), is a real Hilbert-space. If $s : \mathcal{L} \mapsto \mathcal{H}$ is a CH-state on an OML $\mathcal{L}$, it is easy to verify that s, when viewed as a mapping from $\mathcal{L}$ to $\mathcal{H}'$, is a RH-state. Moreover, since for any $x \in \mathcal{H}$, $\langle x, x\rangle = \langle x, x\rangle'$, it is easily seen that if $\mathcal{L}$ admits a strong set of $\mathcal{H}$-valued CH-states, then it admits a strong set of $\mathcal{H}'$-valued RH-states. Conversely, if $\mathcal{H}$ is a real Hilbert-space whose B is a Hilbertian basis, one defines the complexification $\mathcal{H}''$ of $\mathcal{H}$ as being the complex Hilbert-space admitting B as Hilbertian basis. It is easily seen that a $\mathcal{H}$-valued RH-state s on an OML $\mathcal{L}$, when viewed as a mapping from $\mathcal{L}$ to $\mathcal{H}''$, is a $\mathcal{H}''$-valued state. It follows that if $\mathcal{L}$ admits a strong set of $\mathcal{H}$-valued RH-states, then it also admits a strong set of $\mathcal{H}''$-valued CH-states. Moreover, it is not difficult to see, in the same way, that $\mathcal{L}$ admits a strong set of QH-states iff it admits a strong set of RH-states.

If $\mathcal{H}$ is any Hilbert-space, then, for any unit vector $u \in \mathcal{H}$, the mapping $a \mapsto pr_a(u)$ from $\mathcal{C}(\mathcal{H})$ to $\mathcal{H}$ is a H-state s_u on $\mathcal{C}(\mathcal{H})$ such that for any $b \in \mathcal{C}(\mathcal{H})$, $s_u(b) = u$ iff $u \in b$. Since for any $a, b \in \mathcal{C}(\mathcal{H})$ such that $a \nsubseteq b$ there exists a unit vector $u \in a \setminus b$ it follows that the Hilbert lattice $\mathcal{C}(\mathcal{H})$ admits a strong set of H-states, hence, by the remark above, $\mathcal{C}(\mathcal{H})$ admits a strong set of RH-states.

For all these reasons, we will restrict ourselves, in the sequel, to the study of RH-states.

If s is a RH-state on an OML $\mathcal{L}$, then, for any $a \in \mathcal{L}$, $s(a) + s(a^{\perp}) = e_1$, where $e_1 = s(1)$ and $s(a) \perp s(a^{\perp})$, hence $||s(a)||^2 + ||s(a^{\perp})||^2 = 1$, which proves that $||s(a)|| \leq 1$ and that $||s(a)|| = 1 \Leftrightarrow s(a) = e_1$. This also shows that if $s(a) \perp e_1$ then $s(a) = 0$. Moreover, if a, b are any two elements of $\mathcal{L}$ such that $a \perp b$, then $||s(a \vee b)||^2 = ||s(a)||^2 + ||s(b)||^2$, which shows that the mapping $a \mapsto ||s(a)||^2 = \langle s(a), s(a)\rangle$ is a real-valued state on $\mathcal{L}$. It follows also that if $a \leq c$ then $||s(a)|| \leq ||s(c)||$. In particular, if $a \leq c$ and $s(a) = e_1$, then $||s(c)|| = 1$ hence $s(c) = e_1$. It also follows that if $\mathcal{L}$ admits a strong set of RH-states, then $\mathcal{L}$ admits a strong set of real-valued states.

Let us assume that s_1 is a two-valued state on $\mathcal{L}$, that is to say a real-valued state whose only values are 0 and 1. Then, if $\mathcal{H}$ is any real nonzero Hilbert-space, and if e_1 is a unit vector in $\mathcal{H}$, by setting, for any $a \in \mathcal{L}$, $s(a) = s_1(a)e_1$, we obtain a RH-state.

Theorem 5.1. *There is an effective procedure allowing to obtain, from each finite OML $\mathcal{L}$ without a set of RH-states, an equation holding in all OMLs with a strong set of RH-states (hence in particular in all* <u>*classical*</u> *Hilbert lattices), which fails in $\mathcal{L}$.*

Proof. This Theorem can be proved in the same way as Theorem 1, (b) in Mayet (1986), in the particular case of OMLs without strong set of real-valued states, which is illustrated by examples 2 and 3 in Section 8 of Mayet (1986). The procedure obtained here is very similar to the one given in Mayet (1986). Therefore, we will not give the proof of Theorem 5.1, but, as an illustration, we will show how equations E_n can be obtained in this way. □

Let $n \geq 3$ be an integer, and let us consider the OML $\mathcal{L}_n$ whose Greechie diagram is given in Fig. 1.

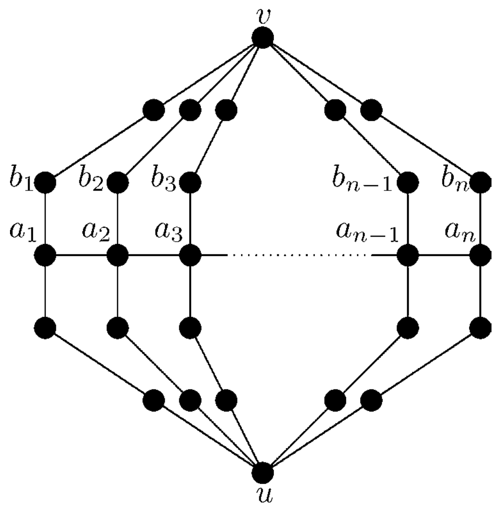


Fig. 1.

Theorem 5.2. *For any $n \geq 3$ the OML $\mathcal{L}_n$ does not admit a strong set of RH-states. The corresponding equation, obtained by Theorem 5.1, which holds in any OML with a strong set of RH-states and fails in $\mathcal{L}_n$, is the equation E_n.*

Proof. Let s be a RH-state on $\mathcal{L}_n$, such that $s(u) = e_1$ where e_1 is a unit vector. Then $e_1 = s(a_1) + \cdots + s(a_n)$, and, for $i = 1, \ldots, n$, $e_1 = s(a_i) + s(b_i)$ (since $(a_i \vee b_i)^\perp \perp u$, hence $s(a_i \vee b_i)^\perp = 0$ and $s(a_i \vee b_i) = e_1$). It follows that $s(a_1) + \cdots + s(a_n) + s(b_1) + \cdots + s(b_n) = ne_1 = e_1 + s(b_1) + \cdots + s(b_n)$, and therefore $s(b_1) + \cdots s(b_n) = (n-1)e_1$. For $i = 1 \cdots n$, we have $v \perp b_i$, thus $s(v) \perp s(b_i)$. It follows that $s(v) \perp s(b_1) + \cdots + s(b_n) = (n-1)e_1$, hence $s(v) = 0$ and $s(v^\perp) = e_1$. Since $u \not\leq v^\perp$, this proves that $\mathcal{L}_n$ does not admit a strong set of RH-states. Let us show that, by using the same procedure as in Mayet (1986) in order to construct from $\mathcal{L}_n$ an equation of $\mathcal{E}$, we obtain equation E_n. In the diagram of Fig. 2 are represented all the hypotheses needed for proving that, for any RH-state s on $\mathcal{L}_n$, $||s(u)|| = 1 \Rightarrow ||s(v^\perp)|| = 1$, with the following understanding:

i) atoms which must be supposed only to be mutually orthogonal are linked together by a dotted line;
ii) if we must use the fact that some atoms are exactly the atoms of a block, these atoms are linked together by a continuous line.

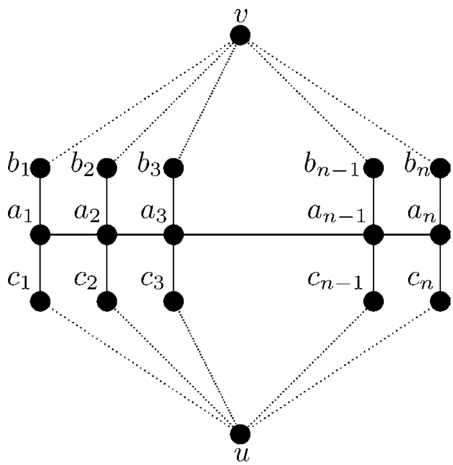


Fig. 2.

The diagram of Fig. 2 means that, if a_i, b_i, c_i (for $i = 1\cdots n$), u and v are any elements of any OML $\mathcal{L}$, such that:

- for $i = 1\cdots n$, a_i, b_i, c_i are mutually orthogonal and $a_i \vee b_i \vee c_i = 1$;
- $a_1, \ldots, a_n$ are mutually orthogonal and $a_1 \vee \cdots \vee a_n = 1$;
- $u \perp c_i$ for $i = 1, \ldots, n$;
- $v \perp b_i$ for $i = 1, \ldots, n$;

then it can be proved that, for any RH-state s on $\mathcal{L}$ satisfying $s(u) = s(1)$, we have $s(v^\perp) = s(1)$. This is easy to verify, since the proof is almost the same as above. The equation corresponding to the above diagram can be written as follows:

$$(\Omega^1) \;\Rightarrow\; u \wedge (a_1 \vee \cdots \vee a_n) \wedge (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_n \vee b_n \vee c_n) \leq v^\perp \quad \left(E_n^1\right)$$

where (Ω^1) is the set of all orthogonality relations appearing in the diagram of Fig. 2. Here, we cannot assume that $a_1 \vee \cdots \vee a_n = a_1 \vee b_1 \vee c_1 = \cdots = a_n \vee b_n \vee c_n = 1$, but we have added these terms on the left-hand side t of the inequality, in order that, for any RH-state s such that $s(t) = s(1)$, we have also $s(a_1 \vee \cdots \vee a_n) = s(a_1 \vee b_1 \vee c_1) = \cdots = s(a_n \vee b_n \vee c_n) = s(1)$, which allows us to prove, in the same way as above, that the equation E_n^1 holds in any OML with a strong set of RH-states.

Equation E_n^1 is not exactly identical to equation E_n, but for obtaining E_n from E_n^1, we need only make some slight modifications. We must delete in (Ω^1) all the conditions of the form $a_i \perp c_i$, $b_i \perp c_i$, $u \perp c_i$ and replace in the inequality c_i by $(a_i \vee b_i)^\perp$, and u by $(a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n)$. Then, all the terms of the form $(a_i \vee b_i \vee c_i)$ on the left-hand side of the inequality must be deleted. Moreover, we must remove in (Ω^1) all the conditions of the form $v \perp b_i$ and replace v by

$b_1^\perp \wedge \cdots \wedge b_n^\perp$. After all these modifications we obtain exactly equation E_n, and it is easy to verify that E_n is equivalent to E_n^1 in the theory of OMLs.

However, let us verify that E_n holds in any OML with a strong set of RH-states. Let $\mathcal{L}$ be any OML with a strong set of RH-states, and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of $\mathcal{L}$ satisfying the conditions of orthogonality (Ω). Let us define $a = a_1 \vee \cdots \vee a_n$, $q = (a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n)$ and $b = b_1 \vee \cdots \vee b_n$. Let s be a RH-state on $\mathcal{L}$ such that $s(a \wedge q) = e_1$, where $e_1 = s(1)$. Then, since $a \wedge q \leq a$, $s(a) = s(a_1) + \cdots + s(a_n) = e_1$, and, for $i = 1, \ldots, n$, since $a \wedge q \leq a_i \vee b_i$, $s(a_i \vee b_i) = s(a_i) + s(b_i) = e_1$. We infer, exactly as above, that $s(b^\perp) = 0$ hence $s(b) = e_1$. Since $\mathcal{L}$ admits a strong set of RH-states, this shows that $a \wedge q \leq b$ thus that equation E_n holds in $\mathcal{L}$.

If $a_1, \cdots a_n, b_1, \ldots, b_n$ are elements of $\mathcal{L}_n$ defined as shown in Fig. 1, then all the conditions (Ω) hold true, and $a \wedge q = u \not\leq v^\perp = b$, therefore E_n fails in $\mathcal{L}_n$. □

We notice that both Theorems 4.1 and 5.2 show that equations E_n belong to $\mathcal{E}$, but, although their proofs are quite similar, they are different: the first one asserts that equations E_n hold in any GHL (except in very particular cases), whereas the second one states that these equations hold in all OMLs with a strong set of RH-states.

Let us denote by $\mathcal{E}_R$ and $\mathcal{E}_{RH}$, the set of all equations in $\mathcal{E}$ which hold in any OML with, respectively, a strong set of real-valued states and a strong set of RH-valued states. Since any OML with a strong set of RH-states admits a strong set of real-valued states, we have the inclusion $\mathcal{E}_R \subseteq \mathcal{E}_{RH}$. We will see hereafter that this inclusion is strict.

Lemma 5.3. *Let k be an integer ≥ 3. For any two atoms $c, d \in \mathcal{L}_k$, such that $c \not\perp d$, $(c, d) \neq (u, v)$ and $(c, d) \neq (v, u)$ (cf. Fig. 3) there exists a two-valued state s on $\mathcal{L}_k$ such that $s(c) = s(d) = 1$.*

Proof. Let us notice first that $\mathcal{L}_k$ (cf. Fig. 3) admits many symmetries (or involutive automorphisms). One of them, σ, is such that $\sigma(u) = v$, and, for $i = 1, \ldots, k, \sigma(u_i) = v_i$ (hence $\sigma(c_i) = c_i$). Other symmetries, denoted by $\sigma_{i,j}$ for $i, j \in \{1, \ldots, k\}$ and $i \langle j$ are characterized by the relations $\sigma_{i,j}(u) = u, \sigma_{i,j}(v) = v$, $\sigma_{i,j}(u_i) = u_j, \sigma_{i,j}(u_l) = u_l$ for any $l \neq i, j$.

Let c, d be two atoms of $\mathcal{L}_k$ such that $c \not\perp d$, $(c, d) \neq (u, v)$ and $(c, d) \neq (v, u)$. Each of the two diagrams in Fig. 4 depict a two-valued state s on $\mathcal{L}_k$, being understood that the black-coloured atoms are exactly atoms x such that $s(x) = 1$, and that, for $i = 2, \ldots, k-2$, $s(u_i) = s(u_1)$, and $s(v_i) = s(v_1)$ (and consequently, similar conditions hold for c_i, $(u \vee u_i)^\perp$, and $(v \vee v_i)^\perp$). By using the above symmetries of $\mathcal{L}_k$ (and other symmetries obtained by composition), it is easy to see that these two states are sufficient to show that there exists a two-valued state s on $\mathcal{L}_k$ such that $s(c) = s(d) = 1$. □

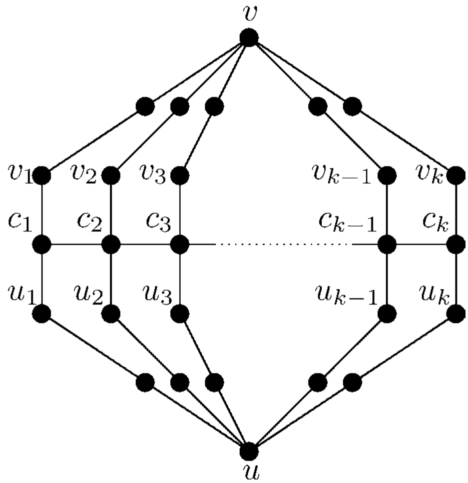


Fig. 3.

Lemma 5.4. *Every OML $\mathcal{L}_k$, $k \geq 3$, admits a strong set of real-valued states.*

Proof. Let x, y be any elements of $\mathcal{L}_k$ such that $x \not\leq y$. It is easy to see that if $(x, y) \neq (u, v^{\perp})$ and $(x, y) \neq (v, u^{\perp})$ there exists two atoms c, d such that $c \leq x$, $d \leq y^{\perp}$, $c \not\perp d$, $(c, d) \neq (u, v)$ and $(c, d) \neq (v, u)$. By Lemma 5.3, there exists a two-valued state s on $\mathcal{L}_k$ such that $s(c) = s(d) = 1$, hence $s(x) = 1$ and $s(y) = 0$.

Let us suppose that $x = u$ and $y = v^{\perp}$, and let s be the real-valued state on $\mathcal{L}_k$ such that (cf. Fig. 3) $s(u) = 1, s(c_1) = \cdots = s(c_k) = \frac{1}{k}, s(v_1) = \cdots = s(v_k) = 1 - \frac{1}{k}$, and $s(v) = \frac{1}{k}$. Then $s(u) = 1$, and $s(v^{\perp}) = 1 - \frac{1}{k} \langle 1$. The case where $x = v$ and $y = u^{\perp}$ is similar. □

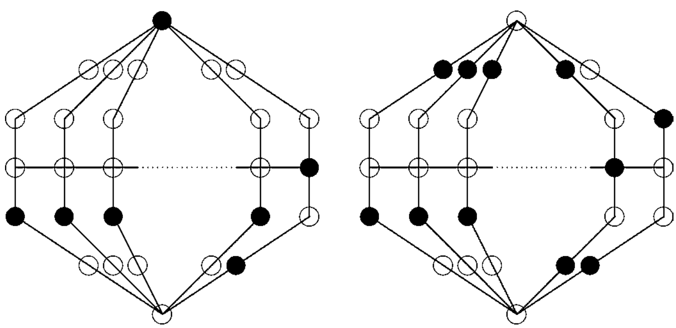


Fig. 4.

Theorem 5.5. *Each equation E_n, $n \geq 3$, is not a consequence of the set of equations $\mathcal{E}_R$ related to strong sets of real-valued states. In particular, $\mathcal{E}_R$ is a proper subset of $\mathcal{E}_{RH}$.*

Proof. We need only use Theorem 5.2 and Lemma 5.4, by which, for any $n \geq 3$, $\mathcal{L}_n$ admits a strong set of real-valued states, whereas equation E_n fails in $\mathcal{L}_n$. □

Lemma 5.6. *Let n be an integer ≥ 3. If $a_1, \ldots a_n, b_1, \ldots b_n$ are any elements of any OML satisfying (Ω), and if s is any real-valued state on $\mathcal{L}$ such that $s((a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n)) = 1$, then $s((b_1 \vee \cdots \vee b_n)^\perp) \leq \frac{1}{n}$.*

Proof. Since $s(a_1) + \cdots + s(a_n) \leq 1$, there exists $i \in \{1, \ldots, n\}$ such that $s(a_i) \leq \frac{1}{n}$. From $s(a_i \vee b_i) = 1$, and $a_i \perp b_i$, it follows that $s(b_i) \geq 1 - \frac{1}{n}$, hence, since $(b_1 \vee \cdots \vee b_n)^\perp \perp b_i$, we have $s((b_1 \vee \cdots \vee b_n)^\perp) \leq \frac{1}{n}$. □

Definition 5.7. Two blocks B, B' of an OML are called adjacent if $B \neq B'$ and $B \cap B' \neq \{0, 1\}$. We will say that an OML $\mathcal{L}$ is plain if any block of $\mathcal{L}$ possesses at least 3 atoms, and, for any two ajacent blocks B, B', their intersection $B \cap B'$ is of the form $\{0, 1, g, g^\perp\}$, where g is an atom of $\mathcal{L}$. It follows from the Loop Lemma (cf. Greechie, 1971; Kalmbach, 1983, that every loop in a plain OML is of order at least 5.

Let us observe that for any $n \geq 3$, $\mathcal{L}_n$ is a plain OML. The following Lemma 5.8 will be useful in the sequel.

Lemma 5.8. *Let $\mathcal{L}$ be a plain OML, let $a_1, \ldots, a_n, b_1, \ldots, b_n$ (where $n \geq 2$) be elements of $\mathcal{L}$ and let us define a, q and b as in Theorem 4.1. Let us assume that (Ω) holds and that $q \leq b$ fails in $\mathcal{L}$. Then q is an atom of $\mathcal{L}$, b is a coatom, and for $i = 1, \ldots, n$ there exists three distinct blocks B_i, B'_i, B''_i such that:*

- *(a) the blocks B_i are pairwise distinct and nonadjacent and, for $i = 1, \ldots, n$, the set of atoms of B_i is $\{a_i, b_i, (a_i \vee b_i)^\perp\}$,*
- *(b) for $i = 1, \ldots, n$, $b_i \in B'_i$ and $(a_i \vee b_i)^\perp \in B''_i$,*
- *(c) for $i \neq j$, $b^\perp$ is the unique atom in $B'_i \cap B'_j$, and q is the unique atom in $B''_i \cap B''_j$.*

Moreover, there exist no real-valued state s on $\mathcal{L}$ such that $s(q) = s(b^\perp) = 1$.

Proof. In this proof, we will often use some (quite obvious) properties of plain OMLs which do not hold in all OMLs.

By (Ω), we have $b_1 \perp a_1 \perp a_2 \perp b_2$ and, if we define $q_0 = (a_1 \vee b_1) \wedge (a_2 \vee b_2)$, $b_0 = b_1 \vee b_2$, then, since $q \not\leq b$, we have $q_0 \not\leq b_0$, hence $q_0 \neq 0$ and $b_0 \neq 1$.

Both a_1 and a_2 are nonzero since, if for instance $a_1 = 0$, then $q_0 = b_1 \wedge (a_2 \vee b_2) \leq b_1 \leq b_0$, a contradiction.

Both b_1 and b_2 are nonzero since, if for instance, $b_1 = 0$, then $q_0 = a_1 \wedge (a_2 \vee b_2)$ hence, by distributivity, $q_0 = a_1 \wedge b_2 \leq b_0$, a contradiction.

We have $a_1 \vee b_1 \neq 1$, since otherwise we would have $b_1 = a_1^{\perp}$, hence $a_2 \leq b_1$, and $q_0 \leq a_2 \vee b_2 \leq b_1 \vee b_2 = b_0$. In the same way, $a_2 \vee b_2 \neq 1$.

If s is a real-valued state on a subOML of $\mathcal{L}$ containing $\{a_1, a_2, b_1, b_2\}$ such that $s(q_0) = 1$, then, since $s(a_1) + s(b_1) = s(a_2) + s(b_2) = 1$ and $s(a_1) + s(a_2) \leq 1$, it follows that $s(b_1) \geq \frac{1}{2}$ or $s(b_2) \geq \frac{1}{2}$, hence $s(b_0) \geq \frac{1}{2}$. Since $q_0 \not\leq b_0$, this implies that any sub-OML of $\mathcal{L}$ containing a_1, a_2, b_1, b_2 does not admit a strong set of two-valued states.

We observe that there exist three blocks B_0, B_1, B_2 of $\mathcal{L}$ such that $\{a_1, a_2\} \subseteq B_0$, $\{a_1, b_1\} \subseteq B_1$, $\{a_2, b_2\} \subseteq B_2$. If a subset M of $\mathcal{L}$ is either a block, or the union of two adjacent blocks, it is easily seen that M is a sub-OML of $\mathcal{L}$ admitting a strong set of two-valued states. It follows, by the above remark about two-valued states, that $b_1 \notin B_0 \cup B_2$ and $b_2 \notin B_0 \cup B_1$, hence in particular the blocks B_0, B_1, B_2 are distinct. Since $a_1 \in (B_0 \cap B_1) \setminus \{0, 1\}$, a_1 is an atom or a coatom, and the same is true for a_2. If a_1 would be a coatom, since $a_1 \perp b_1$ and $b_1 \neq 0$, this would imply $a_1 \vee b_1 = 1$, which is not possible. This proves that both a_1, a_2 are atoms, that a_1 is the unique atom in $B_0 \cap B_1$, and a_2 is the unique atom in $B_0 \cap B_2$. It follows that $B_0 \cap B_1 \cap B_2 = \{0, 1\}$, and therefore that $B_1 \cap B_2 = \{0, 1\}$ since otherwise, (B_0, B_1, B_2) would be a loop of order three.

Let us suppose that b_1 commutes with b_2. Then there exists a block B containing $\{b_1, b_2\}$, and, since $b_1 \notin B_0 \cup B_2$ and $b_2 \notin B_0 \cup B_1$, B is distinct from B_0, B_1, B_2. It follows that $B \cap B_1 = \{b_1, b_1^{\perp}\}$. Since $a_1 \vee b_1 \neq 1$, b_1 is not a coatom, hence it is the unique atom in $B \cap B_1$, and we infer that $B \cap B_1 \cap B_0 = \{0, 1\}$. If $B \cap B_0 \neq \{0, 1\}$, B and B_0 are adjacent, and then (B, B_1, B_0) is a loop of order three. If $B \cap B_0 = \{0, 1\}$, then (B_0, B_1, B, B_2) is a loop of order four. In both cases we obtain a contradiction and therefore b_1 does not commute with b_2

Let B_1' and B_2' be two blocks containing $\{b_0, b_1\}$ and $\{b_0, b_2\}$), respectively. Since b_1 does not commute with b_2, $B_1' \neq B_2'$. Since $B_1 \cap B_2 = \{0, 1\}$, we cannot have $B_1 = B_1'$ together with $B_2 = B_2'$. If $B_2 = B_2'$, we show as above that if $B_1' \cap B_0 \neq \{0, 1\}$, (B_1', B_1, B_0) is a loop of order 3, and otherwise (B_0, B_1, B_1', B_2) is a loop of order 4. This shows that $B_1' \neq B_1$ and $B_2' \neq B_2$. It follows that B_1, B_1' are adjacent, B_2, B_2' too, therefore both b_1 and b_2 are atoms or coatoms. Since $b_0 = b_1 \vee b_2 \neq 1$, and b_1 does not commute with b_2, both b_1 and b_2 are atoms. Since $b_0 \in B_1' \cap B_2'$, and b_1, b_2 are two distinct atoms $\leq b_0$, it follows that b_0 is a coatom, and that $b_0^{\perp}$ is the unique atom in $B_1' \cap B_2'$.

Since $b_0 \leq b \langle 1$, and b_0 is a coatom, it follows that $b = b_0$.

Now, replacing b_1 by $(a_1 \vee b_1)^{\perp}$, b_2 by $(a_2 \vee b_2)^{\perp}$, q_0 by $b_0^{\perp}$, and b_0 by $q_0^{\perp}$, we obtain exactly in the same way that there exists two distinct blocks B_1'', B_2'' such that $(a_1 \vee b_1)^{\perp}$ is the unique atom in $B_1 \cap B_1''$, $(a_2 \vee b_2)^{\perp}$ is the unique atom in

$B_2 \cap B_2''$, $q = q_0$ is the unique atom in $B_1'' \cap B_2''$. This shows in particular that, for $i = 1, 2$, B_i possesses exactly three atoms: a_i, b_i and $(a_i \vee b_i)^\perp$.

To complete the proof of a), b), c) in Lemma 5.8, we need only notice that in the above proofs, we can replace the couple $(1, 2)$ by any couple (i, j), with $i, j \in \{1, \dots, n\}$, and $i \neq j$.

If s_1 is a real-valued state on $\mathcal{L}$ such that $s_1(q) = 1$, then, by the above remark about real-valued states s on sub-OML of $\mathcal{L}$ containing $\{a_1, a_2, b_1, b_2\}$ such that $s(q_0) = 1$, we have $s_1(b) = s_1(b_0) \geq \frac{1}{2}$, hence $s_1(b)^\perp \neq 1$. □

Lemma 5.9. *For any $n \geq 3$ and $k \geq 3$, equation E_n holds in $\mathcal{L}_k$ iff $n \neq k$.*

Proof. Let k and n be two integers ≥ 3, and let $a_1, \dots a_n, b_1, \dots, b_n \in \mathcal{L}_k$, satisfying (Ω). We define a, b, q as in Theorem 4.1. The elements of $\mathcal{L}_k$ will be denoted here as indicated in Fig. 3. If $k \langle n$, there exists $i_0 \in \{1, \dots, n\}$ such that $a_{i_0} = 0$, and then $a \wedge q \leq a_{i_0} \vee b_{i_0} = b_{i_0} \leq b$. If $k = n$, we already know, by Theorem 4.1, that E_n fails in $\mathcal{L}_k$.

Let us assume that $k \rangle n$, and suppose that $q \wedge a \not\leq b$. Then, by Lemma 5.8, q and $b^\perp$ are two atoms of $\mathcal{L}_k$ and there exists no real-valued state s on $\mathcal{L}_k$ such that $s(q) = s(b^\perp) = 1$. Therefore, by Lemma 5.3 and the symmetry σ of $\mathcal{L}_k$, we need only study the case where $q = u$ and $b = v^\perp$.

It follows from Lemma 5.8 that, for $i = 1, \dots, n$, there exists $p \in \{1, \dots, k\}$ such that $b_i = v_p$, $a_i = c_p$ and $(a_i \vee b_i)^\perp = u_p$. The atoms a_i, $i = 1, \dots, n$, being nonzero and mutually orthogonal, they are distinct, and, by the symmetries $\sigma_{i,j}$ of $\mathcal{L}_k$ (cf. the proof of Lemma 5.3), we may assume that, for $i = 1, \dots, n$, $a_i = c_i$ and $b_i = v_i$. Then $a = a_1 \vee \cdots \vee a_n \leq c_k^\perp$, hence $a \wedge q \leq c_k^\perp \wedge u = 0$, a contradiction. □

Theorem 5.10. *For $n \geq 3$, E_n is not a consequence (in the theory of OMLs) of the set of equations $\{E_k : k \geq 3, k \neq n\}$. This still holds in the theory of OMLs satisfying all the equations in $\mathcal{E}_R$.*

Proof. The first part is an obvious consequence of Lemma 5.9. The second part follows from the fact that, by Lemma 5.4, for $k \geq 3$, $\mathcal{L}_k$ admits a strong set of real-valued states, which implies that any equation in $\mathcal{E}_R$ holds in $\mathcal{L}_k$. □

Remark: For each $n \geq 3$ there are some variants of equation E_n. In each of the following equations, Ω, a, b and q are defined as above in Theorem 4.1.
By exchanging the roles of b_i and $(a_i \vee b_i)^\perp$, for $i = 1, \dots, n$, we obtain:

$$(\Omega) \;\Rightarrow\; a \wedge b^\perp \leq q^\perp$$

which can also be written:

$$(\Omega) \;\Rightarrow\; q \leq a^{\perp} \vee b$$

If, in E_{n+1}, we replace a_{n+1} by $a^{\perp} = (a_1 \vee \cdots \vee a_n)^{\perp}$, and b_{n+1} by $a \wedge b$, we obtain:

$$(\Omega) \;\Rightarrow\; q \wedge (a \to b) \leq b.$$

Substituting, in (E_{n+1}), $a^{\perp}$ to a_{n+1} and $\varphi_a(q)$ to b_{n+1} (where φ_a is the Sasaki projection: $\varphi_a(q) = (q \vee a^{\perp}) \wedge a$), we obtain:

$$(\Omega) \;\Rightarrow\; q \leq b \vee \varphi_a(q).$$

We observe that, in both two last cases, the equation obtained is a consequence of E_{n+1}, which fails in $\mathcal{L}_{n+1}$, hence is nontrivial.

6. ANOTHER SEQUENCE OF EQUATIONS

Let $n \geq 2$ be an integer, and let $\mathcal{L}'_n$ be the OML whose Greechie diagram is represented in Fig. 5.

Then $\mathcal{L}'_n$ does not admit a strong set of RH-states. Indeed, let us assume that s is a RH-state on this OML such that $s(u) = s(1) = e_1$, (where $||e_1|| = 1$). Then, for $i = 1, \ldots, n$, $s(a_i) \perp s(b_i)$ and $s(a_i) + s(b_i) = e_1$, hence, since $s(v) \perp s(b_i)$,

$$||s(v)||^2 = \langle s(v), e_1 \rangle = \langle s(v, s(a_i) + s(b_i)\rangle = \langle s(v), s(a_i)\rangle.$$

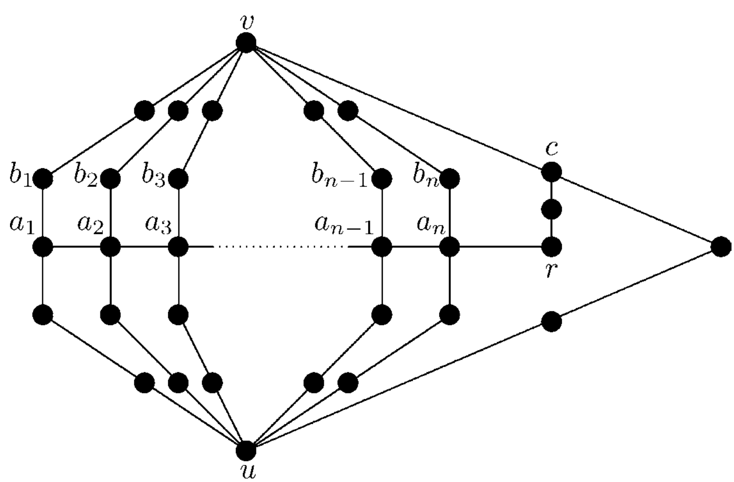


Fig. 5.

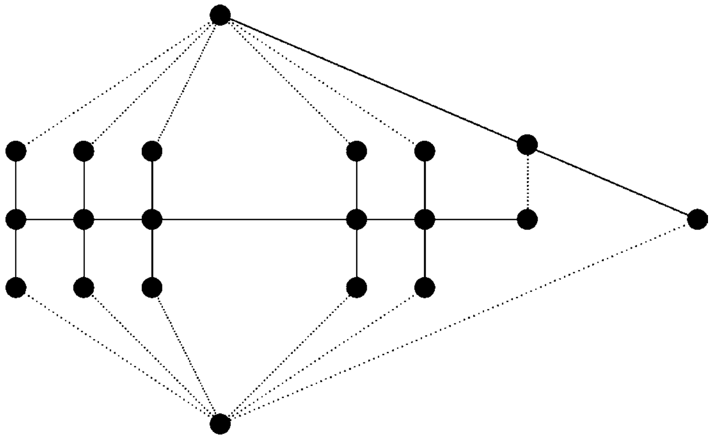


Fig. 6.

Moreover, since $a_1, \ldots, a_n, r$ are mutually orthogonal and $a_1 \vee \cdots \vee a_n \vee r = 1$, $s(a_1) + \cdots + s(a_n) + s(r) = e_1$, hence

$$||s(v)||^2 = \langle s(v), s(a_1) + \cdots + s(a_n) + s(r) \rangle = n||s(v)||^2 + \langle s(v), s(r) \rangle$$

and it follows that $\langle s(v), s(r) \rangle = (1 - n)||s(b^\perp)||^2$.

From $u \leq v \vee c$, it follows that $s(v) + s(c) = e_1$, hence, since $r \perp c$,

$$\langle s(v), s(r) \rangle = \langle s(v) + s(c), s(r) \rangle = \langle e_1, s(r) \rangle = ||s(r)||^2.$$

Therefore, we obtain $||s(r)||^2 + (n - 1)||s(v)||^2 = 0$, and, since each term in this sum is a positive real number, it follows that $s(r) = s(v) = 0$, and $s(v^\perp) = e_1$. Since, $u \not\leq v^\perp$, this proves that $\mathcal{L}'_n$ does not admit a strong set of RH-states. We have also $s(c) = s(v^\perp \wedge r^\perp) = e_1$, whereas $u \not\leq c$. We use the same procedure as in Theorem 5.2 for obtaining an equation in $\mathcal{E}_{RH}$ which fails in $\mathcal{L}'_n$. The hypotheses needed in the above proof are represented in the diagram of Fig. 6. The corresponding equation can be written, after some modifications: ((Ω) and $r \perp a$) $\Rightarrow$ $q \wedge (b \to r^\perp) \wedge (a \vee r) \leq b$, where a, b, q and (Ω) are defined as in Theorem 4.1. It is easy to verify directly that this equation holds in any OML with a strong set of RH-states and fails in $\mathcal{L}'_n$.

Moreover, let us notice that, if we set $r = 0$, we obtain equation E_n, which proves that E'_n is stronger than E_n.

Theorem 6.1. *Let $n \geq 2$ be an integer, let $a_1, \ldots, a_n, b_1, \ldots, b_n, r$ be $2n + 1$ variables, and let (Ω), a, b, q be defined as above in Theorem 4.1. Then the following equation E'_n:*

$$((\Omega) \text{ and } r \perp a) \quad \Rightarrow \quad q \wedge (b \to r^\perp) \wedge (a \vee r) \leq b \qquad (E'_n)$$

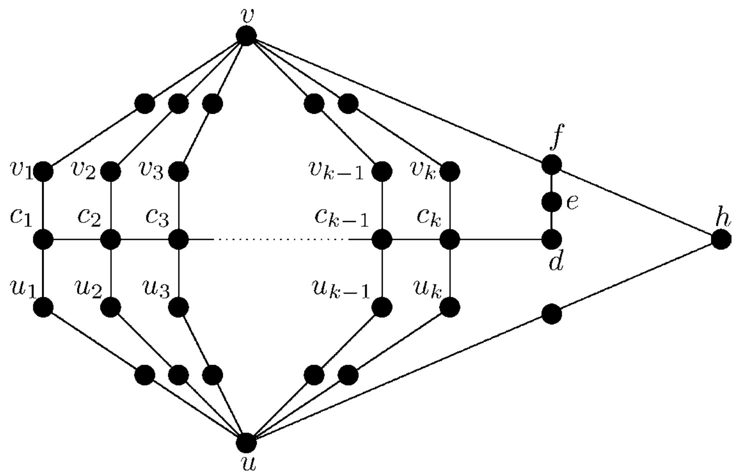


Fig. 7.

holds in any OML with a strong set of RH-states, and fails in the OML $\mathcal{L}'_n$ of Fig. 5. Equation E'_n is stronger than E_n, hence in particular E'_n belongs to $\mathcal{E}_{RH} \setminus \mathcal{E}_R$.

We notice that, if we replace on the right-hand side of the inequality, b by $b \wedge r^{\perp}$, we obtain an equation which is equivalent, since the identity $b \wedge r^{\perp} = b \wedge (b \to r^{\perp})$ holds in any OML.
Substituting $a^{\perp}$ to r in E'_n, and using the identity $(b \to a) \wedge b = a \wedge b$, we obtain:

$$(\Omega) \Rightarrow q \wedge (b \to a) \leq a \wedge b.$$

This equation is nontrivial since it does not hold in $\mathcal{L}'_n$ or in $\mathcal{L}_n$.

Lemma 6.2. *Let k be an integer ≥ 2. For any two atoms g_1, g_2 in $\mathcal{L}'_k$ such that $g_1 \not\perp g_2, \{g_1, g_2\} \neq \{u, d\}, \{u, e\}, \{u, v\}$, (cf. Fig. 5) there exists a two-valued state s on $\mathcal{L}'_k$ such that $s(g_1) = s(g_2) = 1$. But there is no real-valued state on $\mathcal{L}'_k$ such that $s(u) = 1$ together with $s(d) = 1$ or $s(e) = 1$ or $s(v) = 1$.*

Proof. For any pair (i, j) of elements of $\{1, \dots, k\}$ such that $i \neq j$, there exists a unique symmetry (i.e., an involutive automorphism) of $\mathcal{L}'_k$ such that $s(u_i) = u_j$ and $s(u_l) = u_l$ for any $l \neq i, j$.

The two-valued states on $\mathcal{L}'_k$ represented in Fig. 8a, 8b, and 8c, (where it must be understood, in each case, that the black-coloured atoms are exactly atoms x such that $s_m(x) = 1$ and that, for $i = 2, \dots, k-2$, we have $s_m(u_i) = s_m(u_1)$, and $s_m(v_i) = s_m(v_1)$) are sufficient to obtain, modulo the above symmetries, all the needed two-valued states.

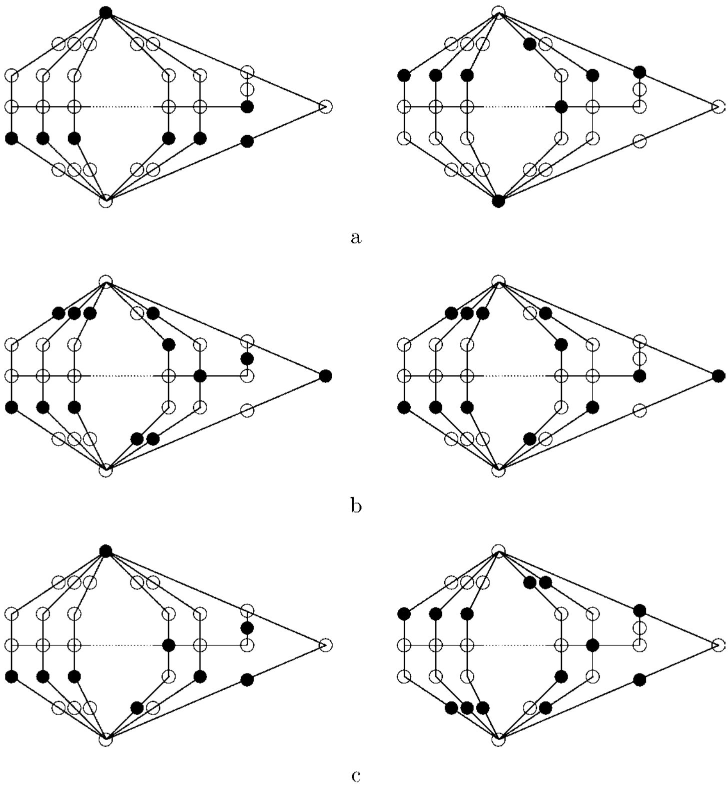


Fig. 8.

Let s be a real-valued state on $\mathcal{L}'_k$ such that $s(u) = 1$
If $s(v) = 1$, then, for $i = 1, \ldots, k$, $s(u_i) = s(v_i) = 0$, hence $s(c_i) = 1$, which contradicts the fact that $s(c_1) + \cdots + s(c_k) = 1$ (with $k \geq 2$).
If $s(d) = 1$ or $s(e) = 1$, then $s(f) = 0$, hence, since $s(h) = 0$, it follows that $s(v) = 1$, which is not possible. □

Lemma 6.3. *Let n, k be two integer ≥ 2. Then E'_n holds in $\mathcal{L}'_k$ iff $n \neq k$.*

Proof. In this proof, the elements of $\mathcal{L}'_k$ are denoted as shown in Fig. 7. We already know that if $k = n$, E'_n fails in $\mathcal{L}'_k$. Let us suppose that $k \neq n$ and that E'_n fails in $\mathcal{L}'_k$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n, r$ be $2n + 1$ elements in $\mathcal{L}'_k$ satisfying the relations (Ω) and $a \perp r$, and such that $q \wedge (b \to r^\perp) \wedge (a \vee r) \not\leq b$, where a, b, q are defined as above.

By Lemma 5.8, both q and $b^{\perp}$ are atoms of $\mathcal{L}'_k$ and this OML admits no real-valued state s such that $s(q) = s(b^{\perp}) = 1$. By Lemma 6.2, we obtain that $\{q, b^{\perp}\}$ is one of the three sets: $\{u, v\}$, $\{u, d\}$, $\{u, e\}$. Moreover, let us observe that, by the part (c) of Lemma 5.8, $b^{\perp}$ belongs to n distinct blocks at least. In the same way, q belongs to n different blocks. It follows that it is not possible that $b^{\perp} = e$ or $q = e$, and it follows that $\{q, b^{\perp}\} = \{u, v\}$ or $\{q, b^{\perp}\} = \{u, d\}$.

a) Let us suppose first that $q = d$, $b^{\perp} = u$.

Then, since d belongs only to two different blocks, by the remark above, $n = 2$. By Lemma 5.8, and the relations of orthogonality $q \perp (a_1 \vee b_1)^{\perp} \perp a_1 \perp b_1 \perp b^{\perp} \perp b_2 \perp a_2 \perp (a_2 \vee b_2)^{\perp} \perp q$, and $a_1 \perp a_2$, it follows that we may suppose (using the fact that a_1, a_2 play symmetrical roles, and $c_1, \ldots c_k$ too) that $q, (a_1 \vee b_1)^{\perp}, a_1, b_1, b^{\perp}, b_2, a_2, (a_2 \vee b_2)^{\perp}$ are equal to $d, c_1, v_1, u_1, u, h, v, f$, respectively. Indeed, under these conditions, we have $a_1 = v_1 \perp v = a_2$. From the assumption $r \perp a$, we obtain that $r \leq (a_1 \vee a_2)^{\perp} = (v_1 \vee v)^{\perp}$. If $r = 0$, then $q \wedge (a \vee r) = d \wedge (v_1 \vee v) = 0$, a contradiction. Otherwise we have $r = (v_1 \vee v)^{\perp}$ hence $q \wedge (b \rightarrow r^{\perp}) = d \wedge (u \vee (u^{\perp} \wedge (v \vee v_1))) = d \wedge u = 0$, another contradiction.

b) Let us assume that $q = u$, $b^{\perp} = d$.

In the same way as in a), we show that $n = 2$, and we may suppose that q, $(a_1 \vee b_1)^{\perp}$, a_1, b_1, $b^{\perp}$, b_2, a_2, $(a_2 \vee b_2)^{\perp}$ are equal to $u, h, v, f, d, , c_1, v_1, u_1$, respectively, and then $r \leq (v \vee v_1)^{\perp}$. It is easily seen that, if $r = 0$, $q \wedge (a \vee r) = 0$, and, if $r = (v \vee v_1)^{\perp}$, $q \wedge (b \rightarrow r^{\perp}) = 0$, hence in both cases, we obtain a contradiction.

c) Now, let us study the case $q = u$, $b^{\perp} = v$.

By Lemma 5.8, for $i = 1, \ldots, n$, there exists a block B_i with 3 atoms $a_i, b_i, (a_i \vee b_i)^{\perp}$, such that $b_i \perp b^{\perp}$ and $(a_i \vee b_i)^{\perp} \perp q$. It follows easily that, for $i = 1, \ldots, n$, there exists $j \in \{1, \ldots, k\}$ such that $a_i = c_j$. Since $a_1, \ldots, a_n$ are distinct, this implies that $n \langle k$, and we may assume, without any loss of generality that, for $i = 1 \cdots, n, a_i = c_i$. If $r \neq a^{\perp}$, then $a \vee r \langle 1$ and $q \wedge (a \vee r) = 0$, a contradiction. Otherwise, $r = a^{\perp}$, and, since $n \langle k$, we have $v^{\perp} \wedge a = 0$, hence $q \wedge (b \rightarrow r^{\perp}) = u \wedge (v^{\perp} \rightarrow a) = u \wedge v = 0$, another contradiction.

d) It remains to study the case $q = v$, $b^{\perp} = u$.

In the same way as in the case c), we show that $n \langle k$ and we may suppose that $a_i = c_i$ for $i = 1, \ldots, n$. Then, if $a \vee r \neq 1$, then $q \wedge (a \vee r) = 0$, a contradiction. Otherwise, $r = a^{\perp}$, thus $b \rightarrow r^{\perp} = u^{\perp} \rightarrow a = u$, hence $q \wedge (b \rightarrow r^{\perp}) = 0$, a contradiction.

This completes the proof of Lemma 6.3. □

Theorem 6.4. *For each $n \geq 2$, the equation E'_n is not a consequence of $\mathcal{E}_0$, and it is not a consequence of $\mathcal{E}_R \cup \{E'_k : k \geq 2$ and $k \neq n\} \cup \{E_k : k \geq 3\}$. In particular, for $n \geq 3$, E'_n is strictly stronger than E_n.*

Proof. Since E'_n is stronger than E_n, it follows, by Theorem 4.2 that E'_n is not a consequence of $\mathcal{E}_0$.

For $j = 1, 2, 3$, there is a unique real-valued state s_j on $\mathcal{L}'_n$ satisfying the following conditions, where the atoms of $\mathcal{L}'_n$ are denoted as in Fig. 7, with $k = n$: $s_1(u) = 1$, $s_1(v) = \frac{2}{2n+1}$, $s_1(d) = s_1(e) = \frac{1}{2n+1}$, $s_1(f) = 1 - \frac{2}{2n+1}$ and, for $i = 1, \ldots, n$, $s_1(c_i) = \frac{2}{2n+1}$ and $s_1(v_i) = 1 - \frac{2}{2n+1}$;
$s_2(e) = s_2(v) = 1, s_2(u) = \frac{1}{n}$ and, for $i = 1, \ldots, n, s_2(c_i) = \frac{1}{n}$ and $s_2(u_i) = 1 - \frac{1}{n}$;
$s_3(d) = 1, s_3(u) = s_3(v) = s_3(h) = \frac{1}{2}$, and, for $i = 1, \ldots, n$, $s_3(u_i) = s_3(v_i) = \frac{1}{2}$.
It is not difficult to see, by using Lemma 6.2, and the states s_1, s_2, s_3, that for any $n \geq 3$, $\mathcal{L}'_n$ admits a strong set of real-valued states, and it follows that any equation in $\mathcal{E}_R$ holds in $\mathcal{L}'_n$.

For any $k \neq n$, by Lemma 6.3, E'_k holds in $\mathcal{L}'_n$, hence E_k too, whereas E'_n fails in $\mathcal{L}'_n$. To complete the proof of Theorem 6.4 we need only show that, for any $n \geq 3$, E_n holds in $\mathcal{L}'_n$. In the same way as in the proof of Lemma 6.3, we prove that, if $a_1, \ldots a_n, b_1, \ldots, b_n$ are elements of $\mathcal{L}'_n$ satisfying (Ω), such that $a \wedge q \not\leq b$ (where a, b, q are defined as above), then we have (cf. Fig. 7, with $k = n$) either $(q, b) = (u, v^\perp)$ or $(q, b) = (v, u^\perp)$ and that in both cases, by the symmetries of $\mathcal{L}'_n$, we may suppose that, for $i = 1 \cdots, n$, $a_i = c_i$. In both cases, we have $a \wedge q = 0$, a contradiction. □

7. SOME OTHER EQUATIONS

In this section, we give many other sequences of equations holding in Hilbert lattices, for which, in some cases, we have not carried out the complete study of the independence relatively to other equations.

Theorem 7.1. *Let $\mathcal{H}$ be an orthomodular space over K, and let $a_1, \ldots a_n, b_1, \ldots b_n \in \mathcal{C}(\mathcal{H})$. Let (Ω), a, b, q be defined as in Theorem 4.1. Then the equation E''_n:*

$$(\Omega) \quad \Rightarrow \quad q \leq b \qquad\qquad (E''_n)$$

holds in $\mathcal{C}(\mathcal{H})$ iff $dim(\mathcal{H})\langle n$ or ($dim(\mathcal{H}) = n$ and $(n-1)1_K \neq 0_K$).

Proof. Let us suppose that the conditions (Ω) and $a \perp r$ hold true.
If $dim(\mathcal{H})\langle n$, there exists $i \in \{1, \ldots, n\}$ such that $a_i = 0$, and it follows that $q \leq b_i \leq b$.
Let us assume that $dim(\mathcal{H}) = n$ and $(n-1)1_K \neq 0_K$. If there exists $i \in \{1, \ldots, n\}$ such that $a_i = 0$, it follows as above that $q \leq b$ holds true. Otherwise, since $dim(a_i) \geq 1$ for $i = 1, \ldots, n$, it follows that $a = \mathcal{H}$, hence, since E_n holds in $\mathcal{C}(\mathcal{H})$, we have $q = a \wedge q \leq b$.
Now, let us suppose that $dim(\mathcal{H}) = n$ and $(n-1)1_K = O_K$. Let us define $a_1, \ldots, a_n, b_1, \ldots, b_n$ as in the part b) of the proof just before Theorem 4.1. Then

$a = \mathcal{H}$, hence $q = a \wedge q$. Since equation E_n fails in $\mathcal{C}(\mathcal{H})$, it follows that E''_n also fails. The last case to study is when $dim(\mathcal{H})\rangle n$. In this case, let $u_1, \ldots, u_{n+1} \in \mathcal{H}$ be $n+1$ pairwise orthogonal vectors. Let us suppose that, for $i = 1, \ldots, n$, a_i and b_i are generated by u_i and by $v_i = \Sigma_{j \neq i} u_j$, respectively. We define $v = \Sigma_{i=1}^{n+1} u_i$. Then $v \in q$ and $v \notin b$. Indeed, let us suppose that there exist $\lambda_1, \ldots, \lambda_n \in K$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$, and let us define $\lambda = \lambda_1 + \cdots + \lambda_n$. Then $v = (\lambda - \lambda_1)u_1 + \cdots + (\lambda - \lambda_n)u_n + \lambda u_{n+1}$, and it follows that $\lambda = 1$, and $\lambda_1 = \cdots = \lambda_n = 0$, a contradiction. This proves that E''_n fails in $\mathcal{C}(\mathcal{H})$. □

Theorem 7.2. *Let $\mathcal{H}$ be an orthomodular space over K, and let $a_1, \ldots a_n, b_1, \ldots b_n, r \in \mathcal{C}(\mathcal{H})$. Let (Ω), a, b, q be defined as in Theorem 4.1. Then the equation E^*_n:*

$$((\Omega) \text{ and } r \perp a) \quad \Rightarrow \quad (a \vee r) \wedge q \leq b \vee r \qquad (E^*_n)$$

holds in $\mathcal{C}(\mathcal{H})$ iff $dim(\mathcal{H})\langle n$ or $(dim(\mathcal{H}) \geq n$ and $(n-1)1_K \neq 0_K)$. For any $n \geq 3$, E_n is a consequence of E^*_n. It follows that $E^*_n \in \mathcal{E}$ and E^*_n is not a consequence of $\mathcal{E}_0$, and is not a consequence of $\mathcal{E}_R \cup \{E_k : k \geq 3, k \neq n\}$.

Proof. If $dim(\mathcal{H})\langle n$, then E''_n holds in $\mathcal{C}(\mathcal{H})$, hence E^*_n too.
Let us assume that $dim(\mathcal{H}) \geq n$ and $(n-1)1_K \neq 0_K$, and let $x \in (a \vee r) \wedge q$. Then, if we define $z = p_r(x)$, and, for $i = 1, \ldots, n$, $x_i = p_{a_i}(x)$, $y_i = p_{b_i}(x)$, we have $x = x_1 + \cdots + x_n + r = x_1 + y_1 = \cdots = x_n + y_n$, hence $(n-1)x = y_1 + \cdots + y_n - z \in b \vee r$. This proves that, if $(n-1)1_K \neq 0_K$, $x \in b \vee r$ hence E^*_n holds true.
By setting $r = 0$ in the equation E^*_n, we obtain equation E_n, and it is easy to complete the proof. □

Remarks:

a) It is easy to prove that, for $n \geq 3$, $E^*_n \in \mathcal{E}_{RH}$.
b) It is not known to us whether or not E^*_n and E_n are equivalent.
c) The equation E^*_n may be written for $n = 2$. We have shown in Section 4 that equation E_2 is trivial, but we do not know whether or not the same is true for E^*_2. This equation, which holds in any GHL, is very simple:

$$a_1 \perp b_1,\ a_2 \perp b_2,\ r \perp a_1 \perp a_2 \perp r \Rightarrow (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_1 \vee a_2 \vee r) \leq b_1 \vee b_2 \vee r \qquad (E^*_2)$$

A simple consequence of equation E^*_2 is:

$$a_1 \perp b_1,\ a_2 \perp b_2,\ a_1 \perp a_2 \quad \Rightarrow \quad (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq b_1 \vee b_2 \vee (a_1 \vee a_2)^{\perp}$$

and we do not know if there exists an OML in which this equation fails.

Theorem 7.3. *Let n_1 and n_2 be two integers such that $n_1 \rangle n_2 \geq 2$. For $k = 1, 2$ and $1 \leq i \leq n_k$, let a_i^k, b_i^k be variables, and let us define $a^k = a_1^k \vee \cdots \vee a_{n_k}^k$, $b^k = b_1^k \vee \cdots \vee b_{n_k}^k$ and $q^k = (a_1^k \vee b_1^k) \wedge \cdots \wedge (a_{n_k}^k \vee b_{n_k}^k)$. Let r be another variable. For $k = 1, 2$, let us denote by (Ω_k) the set of conditions of orthogonality: for $i = 1, \ldots, n_k$, $a_i^k \perp b_i^k$, and for $i, j \in \{1, \ldots, n_k\}$, $i \neq j$, $a_i^k \perp a_j^k$. Let us denote by $E_{(n_1,n_2)}$ the following equation:*

$$((\Omega_1), (\Omega_2), r \perp a^1, r \perp a^2) \Rightarrow (a^1 \vee r) \wedge (a^2 \vee r) \wedge q^1 \wedge q^2 \leq b^1 \vee b^2 \quad (E_{n_1,n_2})$$

Equation E_{n_1,n_2} holds in any GHL whose underlying division ring K satisfies the condition $(n_1 - n_2).1_K \neq 0_K$. In particular, it holds in every classical Hilbert lattice.

Moreover, this equation belongs to $\mathcal{E}_{RH}$ and is not a consequence of $\mathcal{E}_R \cup \{E'_n : n \geq 2\} \cup \{E^*_n : n \geq 2\} \cup \{E_{n'_1,n'_2} : n'_1 \rangle n'_2 \geq 2$ and $(n'_1, n'_2) \neq (n_1, n_2)\}$.

Proof. Let us suppose that r, a_i^k, b_i^k (for $k = 1, 2, i = 1, \ldots, n_k$) are elements of a GHL $\mathcal{L}$ such that the above conditions of orthogonality hold in $\mathcal{L}$, and let $x \in (a^1 \vee r) \wedge (a^2 \vee r) \wedge q^1 \wedge q^2$. Let $z = pr_r(x)$. For $k = 1, 2$ and $1 \leq i \leq n_k$, let us define $x_i^k = pr_{a_i^k}(x)$ and $y_i^k = pr_{b_i^k}(x)$. In the same way as in the proof of Theorem 7.2, we obtain, for $k = 1, 2$, $(n_k - 1)x = y_1^k + \cdots + y_{n_k}^k - z$. By substraction, we have $(n_1 - n_2)x = y_1^1 + \cdots + y_{n_1}^1 - y_1^2 - \cdots - y_{n_2}^2$. It follows that, if $(n_1 - n_2)1_K \neq 0_K$, $x \in b^1 \vee b^2$.

Now, let us suppose that $\mathcal{L}'$ is an OML with a strong set of RH-valued states, and let us prove that E_{n_1,n_2} holds in $\mathcal{L}'$. Let a_i^k, b_i^k (for $k = 1, 2, i = 1, \ldots, n_k$), and r be elements of $\mathcal{L}'$ satisfying the above conditions of orthogonality, and let s be a RH-state on $\mathcal{L}'$ such that $s((a^1 \vee r) \wedge (a^2 \vee r) \wedge q^1 \wedge q^2) = e_1$, with $||e_1|| = 1$. Then it is easy to see that $(n_1 - n_2)e_1 = s(b_1^1) + \cdots + s(b_{n_1}^1) - s(b_1^2) - \cdots - s(b_{n_2}^2)$. Since $(b^1 \vee b^2)^\perp$ is orthogonal to $b_1^1, \ldots, b_{n_1}^1, b_1^2, \ldots, b_{n_2}^2$, it follows that $s((b^1 \vee b^2)^\perp) \perp e_1$, hence $s((b^1 \vee b^2)^\perp) = 0$ and $s(b^1 \vee b^2) = e_1$. Since $\mathcal{L}'$ admits a strong set of RH-valued states, this shows that $(a^1 \vee r) \wedge (a^2 \vee r) \wedge q^1 \wedge q^2 \leq b^1 \vee b^2$.

Let us denote by $\mathcal{L}_{n_1,n_2}$ the OML given in Fig. 9. It is easy to verify, by the same calculations as above, that, for any RH-state s on $\mathcal{L}_{n_1,n_2}$ such that $s(u) = s(1) = e_1$, we have $s(v^\perp) = e_1$. This proves that $\mathcal{L}_{n_1,n_2}$ does not admit a strong set of RH-states. It is not difficult to see that, if we apply Theorem 5.1, we obtain equation E_{n_1,n_2}.

Let us denote by σ the symmetry of $\mathcal{L}_{n_1,n_2}$ such that $\sigma(u) = v$ and $\sigma(c_i^k) = c_i^k$ for $k = 1, 2$ and $1 \leq i \leq n_k$. For $k = 1, 2$, and $1 \leq i \langle j \leq n_k$, let $\sigma_{(k,i,j)}$ be the symmetry of $\mathcal{L}_{n_1,n_2}$ such that $\sigma_{(k,i,j)}(c_i^k) = c_j^k$, $\sigma_{(k,i,j)}(u) = u$, $\sigma_{(k,i,j)}(v) = v$, and $\sigma_{(k,i,j)}(c_{i'}^{k'}) = c_{i'}^{k'}$ for $(k', i') \neq (k, i), (k, j)$. The 10(a) and 10(b) represent two-valued states on $\mathcal{L}_{n_1,n_2}$, with the same understanding as above in Figures 4, 8(a), 8(b), 8(c). These four states are sufficient to see, by using the above symmetries σ and $\sigma_{(k,i,j)}$, that for any two atoms f, g of $\mathcal{L}_{n_1,n_2}$ such that $f \not\perp g$ and $\{f, g\} \neq$

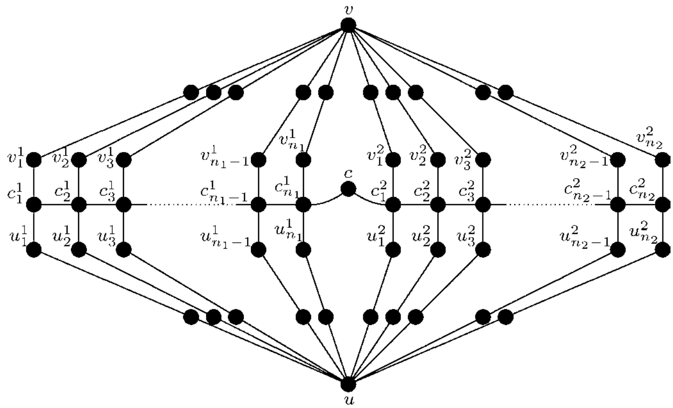
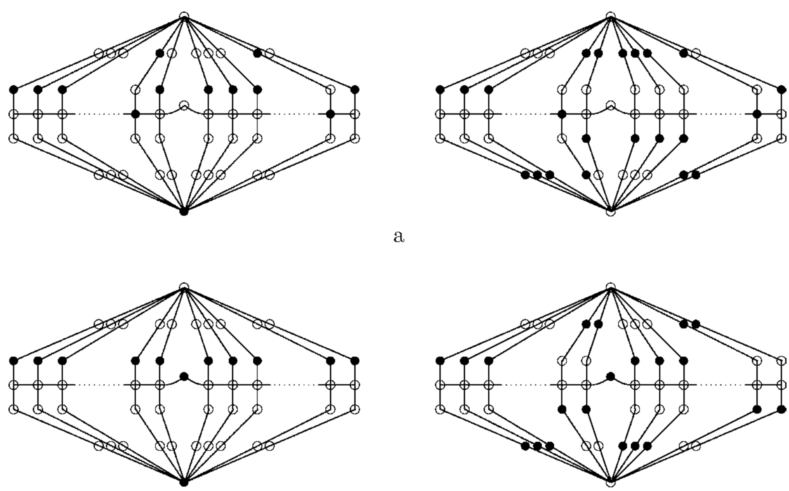


Fig. 9.

$\{u, v\}$, there exists a two-valued state s on $\mathcal{L}_{n_1,n_2}$ such that $s(f) = s(g) = 1$. It follows that if d, e are any two elements of $\mathcal{L}_{n_1,n_2}$ such that $d \not\leq e$, $(d, e) \neq (u, v^{\perp})$ and $(d, e) \neq (v, u^{\perp})$, there exists a two-valued state s on $\mathcal{L}_{n_1,n_2}$ such that $s(d) = 1$ and $s(e) = 0$.

Moreover, if s is any real-valued state on $\mathcal{L}_{n_1,n_2}$ such that $s(u) = 1$, since $s(c^1_1) + \cdots + s(c^1_{n_1}) \leq 1$, and $n_1 \geq 3$, there exists i such that $s(c^1_i) \leq \frac{1}{3}$, and it follows that $s(v^1_i) \geq \frac{2}{3}$, hence $s(v) \leq \frac{1}{3}$.



a

b

Fig. 10.

It is easy to see that there is a unique real-valued state s on $\mathcal{L}_{n_1,n_2}$ such that $s(u) = 1$, for $i = 1, \dots, n_1$, $s(c_i^1) = \frac{1}{n_1}$, for $j = 1, \dots, n_2$, $s(c_j^2) = \frac{1}{n_2}$ and $s(v) = \frac{1}{n_1}$, hence $s(v^\perp) = 1 - \frac{1}{n_1}$. This proves (by the symmetry σ) that $\mathcal{L}_{n_1,n_2}$ admits a strong set of real-valued states.

Let $n \geq 2$ be an integer, and let us suppose that E'_n fails in $\mathcal{L}_{n_1,n_2}$.

Let $a_1, \dots, a_n, b_1, \dots, b_n, r$ be elements of $\mathcal{L}_{n_1,n_2}$ satisfying conditions (Ω) and $r \perp a$, and such that $q \wedge (b \to r^\perp) \wedge (a \vee r) \not\leq b$, where a, b, q and (Ω) are defined as in Theorem 4.1. Then, by Lemma 5.8, for $i = 1, \dots, n$, a_i, b_i and $(a_i \vee b_i)^\perp$ are atoms of $\mathcal{L}_{n_1,n_2}$. Moreover, by considering real-valued states, we have necessarily $(q, b) = (u, v^\perp)$ or $(q, b) = (v, u^\perp)$, and, by the symmetry σ, we may (and do) assume that $(q, b) = (u, v^\perp)$. For $i = 1, \dots, n$, since $a_i, b_i, (a_i \vee b_i)^\perp$ all belong to a same block, and by the relations $b_i \leq b$ and $q \leq (a_i \vee b_i)^\perp$, it follows that there exists $k \in \{1, 2\}$ and j, with $1 \leq j \leq n_k$, such that $a_i = c_j^k$ and $b_i = v_j^k$ (cf. Fig. 9). Since $a_1, \dots, a_n$ are atoms pairwise orthogonal, k does not depend on i and necessarily we have $n \leq n_k$. We will assume, for instance, that $k = 1$, and, by the symmetries $\sigma_{1,i,j}$, we may (and do) suppose that for $i = 1, \dots, n$, $a_i = c_i^1$ and $b_i = v_i^1$. If $a \vee r \langle 1$, since $a \vee r$ belongs to the block B whose atoms are $c_1^1, \dots, c_{n_1}^1, c$, it follows that $q \wedge (a \vee r) = 0$, a contradiction. Otherwise, $r = a^\perp$, thus, since $a \in B$ and $a \neq 1$, we have $b \wedge r^\perp = b \wedge a = 0$ hence $b \to r^\perp = b^\perp$ and $q \wedge (b \to r^\perp) = 0$, another contradiction. This show that, for $n \geq 2$, E'_n holds in $\mathcal{L}_{n_1,n_2}$.

Let $n \geq 2$, and let us assume that E_n^* fails in $\mathcal{L}_{n_1,n_2}$. Let $a_1, \dots, a_n, b_1, \dots, b_n, r$ be elements of $\mathcal{L}_{n_1,n_2}$ satisfying conditions (Ω) and $r \perp a$, such that $(a \vee r) \wedge q \not\leq b \vee r$, where (Ω), a, b, q are defined as in Theorem 7.2. Since $q \not\leq b$, we may apply Lemma 5.8, and, in the same way as above, we may suppose, without any loss of generality, that $q = u$, $b = v^\perp$, $n \leq n_1$, and for $i = 1, \dots, n$, $a_i = c_i^1$ and $b_i = v_i^1$. If $a \vee r \langle 1$, then $(a \vee r) \wedge q = 0$, a contradiction. If $a \vee r = 1$ then, since $c \leq r$, we have $b \vee r = 1$, another contradiction.

Let (n'_1, n'_2) such that $n'_1 \rangle n'_2 \geq 2$ and $(n'_1, n'_2) \neq (n_1, n_2)$, and let us prove that $E_{n'_1,n'_2}$ holds in $\mathcal{L}_{n_1,n_2}$. Let us suppose that this equation fails in $\mathcal{L}_{n_1,n_2}$, and let $a_1^k, \dots, a_{n'_k}^k, b^k, \dots, b_{n'_k}^k$, $k = 1, 2$, and r be elements of $\mathcal{L}_{n_1,n_2}$ satisfying conditions (Ω_1), (Ω_2), $r \perp a^1$, $r \perp a^2$, and such that $(a^1 \vee r) \wedge (a^2 \vee r) \wedge q^1 \wedge q^2 \not\leq b^1 \vee b^2$ (where it is assumed that, in these conditions, n_1, n_2 are replaced by n'_1, n'_2, respectively). Since (Ω_1) holds and $q_1 \leq b$ fails in $\mathcal{L}_{n_1,n_2}$, by Lemma 5.8, and by the above result about real-valued states on $\mathcal{L}_{n_1,n_2}$, it follows that (q^1, b^1) is either $(u, v^\perp)$, or $(v, u^\perp)$ (see Fig. 9) and the same is true for (q^2, b^2). We cannot have, for instance, $q^1 = u$ and $q^2 = v$ since then $q^1 \wedge q^2 = 0$, a contradiction. Therefore, by the symmetry σ of $\mathcal{L}_{n_1,n_2}$, we may assume that $q^1 = q^2 = u$ and $b^1 = b^2 = v^\perp$.

By Lemma 5.8, there exists $k_1 \in \{1, 2\}$ such that for $i = 1, \dots, n'_1$ there exists $i' = f_1(i) \in \{1, \dots, n_{k_1}\}$ such that $a_i^1 = c_{i'}^{k_1}$. In the same way, there exists $k_2 \in \{1, 2\}$ such that each element a_i^2 is of the form $c_{i'}^{k_2}$, where $i' = f_2(i) \in \{1, \dots, n_{k_2}\}$.

Since the mappings f_1 and f_2 are injective, it follows that $n'_1 \leq n_{k_1}$ and $n'_2 \leq n_{k_2}$. Moreover, since $r \perp a^1$ and $r \perp a_2$, it follows that $r \leq c$.

If $n'_1 \langle n_{k_1}$, there exists an integer $i' \in \{1, \ldots, n_{k_1}\}$ which is not of the form $f_1(i)$. Then $a^1 \vee r \leq (c^{k_1}_{i'})^\perp$, thus $(a^1 \vee r) \wedge q^1 = 0$, a contradiction. Therefore, $n'_1 = n_{k_1}$, and, in the same way, we show that $n'_2 = n_{k_2}$. Since $n_1 \rangle n_2$ and $n'_1 \rangle n'_2$, we obtain $(n'_1, n'_2) = (n_1, n_2)$, a contradiction.

Since $\mathcal{L}_{n_1,n_2}$ admits a strong set of real-valued states, it satisfies all the equations in $\mathcal{E}_R$. We have shown above that each equation of $\{E'_n : n \geq 2\} \cup \{E^*_n : n \geq 2\} \cup \{E_{n'_1,n'_2} : n'_1 \rangle n'_2 \geq 2, (n'_1, n'_2) \neq (n_1, n_2)\}$ holds in $\mathcal{L}_{n_1,n_2}$. Since E_{n_1,n_2} fails in $\mathcal{L}_{n_1,n_2}$, this completes the proof of Theorem 7.3. □

In the following Theorems 7.4 and 7.5 we give other sequences of equations holding in all HLs (even in most GHL) for which we have not carried out the complete study.

In these two Theorems, we assume that m is an integer ≥ 2 and that to each integer $k \in \{1, \ldots, m\}$, are associated an integer $n_k \geq 2$ and $2n_k$ variables $a^k_1, \ldots, a^k_{n_k}, b^k_1, \ldots b^k_{n_k}$. We define $a^k = a^k_1 \vee \cdots \vee a^k_{n_k}$, $b^k = b^k_1 \vee \cdots \vee b^k_{n_k}$, and $q^k = (a^k_1 \vee b^k_1) \wedge \cdots \wedge (a^k_{n_k} \vee b^k_{n_k})$. Moreover, we denote by (Ω_k) the set of conditions of orthogonality: for $i = 1, \ldots, n_k$, $a^k_i \perp b^k_i$, and for $i, j \in \{1, \ldots, n_k\}$, $i \neq j$, $a^k_i \perp a^k_j$. In both cases, we denote by $\mathcal{L}$ a GHL and by K its underlying division ring. Other variables will be denoted by r_j (where j is an integer).

It is easy to see, in both cases, that the equations belong to $\mathcal{E}_{RH}$. Each of them may be obtained by applying Theorem 5.1 to a plain OML in which the equation fails. This OML is obtained by pasting copies of the OMLs $\mathcal{L}_{n_k}$ (cf. Fig. 1) modified by adding one or two atoms to their main block (the main block of the OML $\mathcal{L}_n$ of Fig. 1 being this one whose atoms are $a_1, \ldots, a_n$), and possibly by adding some new blocks containing these new atoms. These OMLs will not be depicted but each of them can easily been constructed, leaving oneself be guided by the corresponding equation.

Moreover, it is not difficult to obtain results about the independence of the equations obtained in Theorem 7.4, by using the same methods as in the proof of Theorem 7.3.

Theorem 7.4. *Let us assume that* $(\sum_{k=1}^{m} (-1)^k n_k).1_K \neq 0_K$.

If we define $a^* = (a^1 \vee r_1) \wedge (r_1 \vee a^2 \vee r_2) \wedge \cdots \wedge (r_{m-2} \vee a^{m-1} \vee r_{m-1}) \wedge (r_{m-1} \vee a^m)$*, then the following equation holds in* $\mathcal{L}$*:*

$$(\Omega_1), \ldots, (\Omega_m),\ a^1 \perp r_1 \perp a^2 \perp r_2 \perp \cdots \perp a^{m-1} \perp r_{m-1} \perp a^m$$
$$\Rightarrow a^* \wedge q^1 \wedge \cdots \wedge q^m \leq b^1 \vee \cdots \vee b^m.$$

Moreover, if m is an even number, $m \geq 6$*, and if we define* $a'^* = (r_m \vee a^1 \vee r_1) \wedge (r_1 \vee a^2 \vee r_2) \wedge \cdots \wedge (r_{m-1} \vee a^m \vee r_m)$*, the following equation holds in* $\mathcal{L}$*:*

$$(\Omega_1), \ldots, (\Omega_m), \ r_m \perp a^1 \perp r_1 \perp a^2 \perp r_2 \perp \cdots \perp a^m \perp r_m$$
$$\Rightarrow a'^* \wedge q^1 \wedge \cdots \wedge q^m \ \leq \ b^1 \vee \cdots \vee b^m.$$

Proof. The proof is almost the same as the above proof of Theorem 7.3. We need only use, instead of the substraction of two equalities, the linear combination of m equalities, with the coefficients $+1$ and -1 alternately. □

Theorem 7.5. *Let us assume that the integer m (see above, before Theorem 7.4) is* ≥ 3*. Let us denote by* (Ω) *the set of conditions of orthogonality:* $\{r_i \perp r_j : i, j \in \{1, \ldots, m\}, i \neq j\}$*. Let us define* $r^* = r_1 \vee \cdots \vee r_m$*, and* $a^* = (a^1 \vee r_1) \wedge \cdots \wedge (a^m \vee r_m)$*. Let us assume that* $(1 - m + \sum_{k=1}^{m} n_k)1_K \neq 0_K$*. Then the following equation holds in* $\mathcal{L}$*:*

$$(\Omega), \ (\Omega_1), \ldots, (\Omega_m), \ a^1 \perp r_1, \ldots, \ a^m \perp r_m \Rightarrow$$
$$a^* \wedge r^* \wedge q^1 \wedge \cdots \wedge q^m \ \leq \ b^1 \vee \cdots \vee b^m.$$

Proof. Let us assume that a_i^k, b_i^k, r_k (for $k = 1, \ldots, m$, $i = 1, \ldots, n_k$) are elements of $\mathcal{L}$ satisfying the above conditions of orthogonality, and let $x \in a^* \wedge r^* \wedge q^1 \wedge \cdots \wedge q^m$. Then, for $k = 1, \ldots, m$, we show, in the same way as in the proof of Theorem 7.2, that $(n_k - 1)x = y_1^k + \cdots + y_{n_k}^k - z_k$, where $y_i^k = pr_{b_i^k}(x)$, and $z_k = pr_{r_k}(x)$. Since $x = \sum_{k=1}^{m} z_k$, it follows, by summation, that $(1 - m + \sum_{k=1}^{m} n_k)x \in b^1 \vee \cdots \vee b^m$, hence $x \in b^1 \vee \cdots \vee b^m$. □

8. CONCLUDING REMARKS

It is not very difficult to obtain other equations in the same way. For instance, it is possible to give a (quite complicated) general result including all equations obtained in Theorems 7.2, 7.4 and 7.5, and many other ones. But it seems difficult to obtain in this way a simple equational basis (if it does exist) of the variety generated by the class of HLs. From this viewpoint, it would be interesting to explore new ways allowing to obtain equations holding in HLs. Hereafter, we show that it is possible to obtain an equation in $\mathcal{E} \setminus \mathcal{E}_R$ by using the method of the strong set of real-valued states together with a tensorial product. Unfortunately the equation obtained belongs to $\mathcal{E}_{RH}$ and, more precisely, is a consequence of the equation E_2' studied above in Section 6. The starting point of this method is the example, due to Foulis and Randall, given in Kalmbach (1983), 265, of a finite OML L such that the tensorial product $L \otimes L$ does not exist.

We intend to show that the following equation (where a, b, q, (Ω) are defined as in Propositon 6.1, with n=2) belongs to $\mathcal{E} \setminus \mathcal{E}_R$:

$$((\Omega)\,,\; r \perp a) \Rightarrow q \wedge (b \to r^{\perp}) \wedge (a \vee r) \leq r^{\perp} \to b.$$

Let us suppose that a_1, a_2, b_1, b_2, r are elements of a (classical) Hilbert lattice $\mathcal{L}$, satisfying the conditions (Ω) and $r \perp a$. Let us prove that the inequality $q' \leq r^{\perp} \to b$, where $q' = q \wedge (b \to r^{\perp}) \wedge (a \vee r)$, holds in $\mathcal{L}$. Since $\mathcal{L}$ is a Hilbert lattice the tensorial product $\mathcal{L}^* = \mathcal{L} \otimes \mathcal{L}$ (in the sense of Foulis-Randall, cf. Kalmbach, 1983, p. 264) exists. The OML $\mathcal{L}^*$ contains all the elements of the form $u \otimes v$, for any $u, v \in \mathcal{L}$, and, by definition, $u \otimes v \perp u' \otimes v'$ iff $u \perp u'$ or $v \perp v'$. Moreover, for any real-valued state s on $\mathcal{L}$, there exists a real-valued state s^* on $\mathcal{L}^*$ such that for any $u, v \in \mathcal{L}$, $s^*(u \otimes v) = s(u)s(v)$.

Let s be a real-valued state on $\mathcal{L}$ such that $s(q') = 1$. Let us define $\alpha = s(a_1)$, $\beta = s(a_2)$ and $\gamma = s(b^{\perp})$. Then, from $s(q) = 1$, it follows $s(a_1 \vee b_1) = s(a_2 \vee b_2) = 1$, thus $s(b_1) = 1 - \alpha$ and $s(b_2) = 1 - \beta$. Observing that $t = (b^{\perp} \otimes b^{\perp}) \vee (b_1 \otimes a_1) \vee (a_1 \otimes b_2) \vee (b_2 \otimes a_2) \vee (a_2 \otimes b_1)$ is the supremum, in $\mathcal{L}^*$, of mutually orthogonal elements, we obtain: $s^*(t) = \gamma^2 + (1-\alpha)\alpha + \alpha(1-\beta) + (1-\beta)\beta + \beta(1-\alpha) = \gamma^2 + 1 - (1-(\alpha+\beta))^2 = 1 + s(b^{\perp})^2 - s(r)^2$. Since s^* is a state on $\mathcal{L}^*$, we have $s^*(t) \leq 1$, hence $s(b^{\perp}) \leq s(r)$. Since $s(b \to r^{\perp}) = s(b^{\perp}) + s(b \wedge r^{\perp}) = 1$, it follows that $s(b \wedge r^{\perp}) + s(r) = s(r^{\perp} \to b) = 1$. Since $\mathcal{L}$ admits a strong set of real-valued states, we conclude that $q' \leq r^{\perp} \to b$, hence the above equation holds in $\mathcal{L}$.

If we replace in equation E_2', (cf. the remark after Theorem 6.1) on the left-hand side of the inequality, b by $b \wedge r^{\perp}$, the new equation obtained is equivalent to E_2', hence, since $b \wedge r^{\perp} \leq r^{\perp} \to b$, it follows that the equation above is a consequence of E_2', hence belongs to $\mathcal{E}_{RH}$.

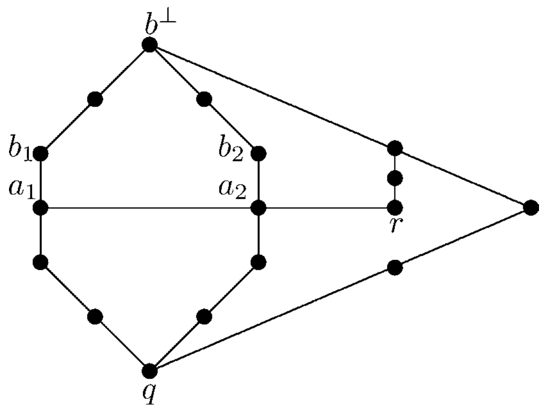


Fig. 11.

This equation does not belong to $\mathcal{E}_R$. Indeed, it is easy to verify that it fails in the OML L'_2 (cf. Fig. 11), and we have proved that this OML admits a strong set of real-valued states.

So, the equation obtained in this case in quite disappointing. It is not known to us whether or not this method using the tensorial product is liable to produce new equations.

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